Explicit Local Class Field Theory à la Lubin and Tate with an Application to Algebraic Topology

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Abstract

Local class field theory was originally proved via global class field theory, and there was no explicit description of the local Artin map and the maximal abelian extension K^{ab} of a local field K. In 1965, Lubin and Tate constructed an explicit form of the local Artin map and K^{ab} from formal group laws. In 1979, Coleman proved an interpolation theorem on division values in local fields by constructing a norm operator depending on Lubin-Tate formal group laws. On the other hand, in topology, Ando established an algebraic criterion on when a complex orientation MU $\to E_n$ for Morava E-theory is an H_∞-map. The criterion relates desired orientations to a specific property of formal group laws.

This thesis has two parts. Firstly, we prove explicit local class field theory following of Lubin and Tate. Secondly, we give a new proof of Ando's theorem in topology via Coleman's norm operator from explicit local class field theory.

Keywords: local class field theory, Lubin-Tate formal group law, Coleman norm operator, Morava E-theory, complex orientation

Contents

1 Introduction

1.1 Local Class Field Theory

The motivation of class field theory is to generate all the Galois extensions of a field from the information of the field itself. In particular, local class field theory wants to generate all the Galois extensions of a local field.

Historically, local class field theory arises from a problem proposed by Emil Artin(1929) that whether one can generalize the norm residue symbol to arbitrary fields that do not contain n-th roots of unity [FLR14]. Helmut Hasse(1930) solved this problem using the global Artin reciprocity law. For an abelian extension L/K (K, L may not be local fields), $\alpha \in K^*$ and v a place of K, the generalized norm residue symbol $(\alpha, L/K)_v$ is an element in the decomposition group of any $w \mid v$ [Con]. It is an analogy of Hilbert's symbol. The precise definition of the norm-residue symbol requires global class field theory. This led Hasse to the discovery of local class field theory. We first need a lemma to see this.

Lemma 1.1. *Suppose F/K_v is a finite field extension for some number field K and a finite place v of K, where K_v is the completion of K with respect to v. Then there exists a number field L/K such that $F = LK_v$, $[L : K] = [F : K_v]$ and $F = L_w$ for some place w of L extending v.*

Proof. Suppose $F = K_v(\alpha)$ and $f \in K_v[X]$ is the minimal polynomial of α over K_v. By [Hu21, Corollary 3.2.16], there is a separable and irreducible polynomial $g \in K[X]$ close enough to f with $\deg(g) = \deg(f)$ such that $K_v(\beta) = F$ for some root β of g. Then $[F : K_v] = \deg(f) = \deg(g) = [K(\beta) : K]$. Since F is a finite extension of a complete field K_v, F is itself complete. Since $F \supset L := K(\beta)$, F is a completion of L with respect to some valuation w of L. $\square$

Here is how local class field theory shows up: Given an abelian extension F/K_v, there exists a field extension L/K such that $F = LK_v$, $[L : K] = [F : K_v]$ and $F = L_w$ for some place w of L extending v by the lemma. Thus, $\mathrm{Gal}(F/K_v) \cong \mathrm{Gal}(L_w/K_v)$. Note that there is a natural inclusion $\mathrm{Gal}(L_w/K_v) \to \mathrm{Gal}(L/K)$ by $\sigma \mapsto \sigma|_L$, mapping $\mathrm{Gal}(L_w/K_v)$ to the decomposition group of $w \mid v$. For any $\alpha \in K^*$, let $(\alpha, F/K_v)$ be the image of $(\alpha, L/K)_v$ in

$\text{Gal}(F/K_v)$. Therefore, we get a homomorphism

$$K^* \to \text{Gal}(F/K_v) \quad \alpha \mapsto (\alpha, F/K_v)$$

The definition of $(\alpha, L/K)_v$ implies that $(\alpha, L/K)_v = Id$ when $v(\alpha)$ is large enough [Con]. Thus, the above map can be extended to $K_v^* \mapsto \text{Gal}(F/K_v)$, which is now called the local Artin map.

As discussed above, local class field theory is derived from the global class field theory originally and there is no explicit description of the local Artin map. The significance of the proof by Lubin and Tate is to give an explicit description of the local Artin map and the maximal abelian extension K^{ab}.

1.2 Relationship between Local Class Field Theory and Algebraic Topology

An important tool used in Lubin and Tate's proof is the Lubin-Tate formal group law. Suppose a prime number p is an uniformizer of the local field, i.e., the local field is an unramified extension of $\mathbb{Q}_p$. Then Lubin-Tate formal group law reduces to a Honda formal group law over the residue field, whose p-series is of the form T^{p^n} for some positive integer n. In 1979, Coleman [Col79] proved an interpolation theorem on division values in local fields by constructing a norm operator $\mathscr{N}_F$ depending on Lubin-Tate formal group law F such that

$$\mathscr{N}_F(g) \circ [p]_F(T) = \prod_{\lambda \text{ is a root of } [p]_F} g \circ F(T, \lambda)$$

where $[p]_F$ is the p-series of F.

On the other hand, there is a series of significant complex oriented spectra in algebraic topology called Morava E-theories E_n, whose coefficient ring $(E_n)_*$ classifies deformations of a formal group law of height n over some perfect field of characteristic p to some complete local ring R. Morava E-theories carry important structure on the cohomology theory called power operation (cf. [GH04, Corollary 7.6] and [BMMS86]). Suppose MU is the complex cobordism theory. It is well-known that MU admits power operation as well(cf. [May77, §IV.2] and [BMMS86]). We also know that a complex orientation on E_n is same to a map between ring spectra MU $\to E_n$. Ando [And95, Theorem 4] gave a criterion about when power operations on MU and E_n are compatible under such a map in terms of the formal group law F associated to the complex orientation in the case that $(E_n)_*$ classifies the deformation of a Honda formal group law. The formal group law satisfies the criterion if

$$[p]_F(T) = \prod_{\lambda \text{ is a root of } [p]_F} F(T, \lambda)$$

Rezk conjectured that the norm operator and Ando's theorem are closely related.

Following Rezk's idea, we will prove Ando's theorem via Coleman norm operator. The original definition of the norm operator only applies to the special case when R is a complete DVR with uniformizer p. Therefore, we will generalize the definition of the norm operator to complete local domain with $p \neq 0$. In particular, $(E_n)_*$ satisfies such conditions.

1.3 Outline of the Thesis

Section 2 will prove the main theorems of local class field theory via Lubin-Tate formal group law.

Section 3 is a quick introduction to Ando's theorem. We will omit most details and only provide necessary background knowledge of the theorem.

Finally, Section 4 is the proof of Ando's theorem via Coleman norm operator.

Section 2 and Section 3 are separate parts in algebraic number theory and algebraic topology respectively. To understand Ando's theorem in Section 4, one needs knowledge from Section 3. The proof of Ando's theorem is based on Subsection 2.2 and part of Subsection 2.3.

2 Local Class Field Theory and Proof by Lubin-Tate Formal Group Laws

2.1 Statements of Main Theorems

By a local field, we mean a field K that is one of the following cases:

1. $K = \mathbb{R}$ or $K = \mathbb{C}$ with the usual absolute value.

2. K is complete with respect to a discrete valuation whose valuation ring has finite residue field.

By [Hu21, Proposition 4.1.4], the latter case is either a finite extension of $\mathbb{Q}_p$ or a finite extension of $\mathbb{F}_p(\!(T)\!)$. The former one is called **archimedean** while the latter case is called **non-archimedean**.

Let K be a local field and $K^{al} \supset K^{ab} \supset K^{un}$ be its algebraic, separable and abelian closure respectively. Let $\mathcal{O}_K$ be the integer ring of K and $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_K$ and $k = \mathcal{O}_K/\mathfrak{m}$ is the residue field with q elements, where q is a power of a prime number p. Suppose L/K is a finite extension, $\mathrm{Nm}_{L/K}(x)$ is the norm of $x \in L$ with respect to L/K.

Let $\mathrm{Gal}(K^{ab}/K)$ be the Galois group of K^{ab}/K. We assign Krull topology to $\mathrm{Gal}(K^{ab}/K)$, i.e., $\mathrm{Gal}(K^{ab}/E)$ forms a fundamental system of neighborhoods of 1 in $\mathrm{Gal}(K^{ab}/K)$, where E runs through all finite abelian extensions of K.

The main theorems of the abelian local class field theory are the following:

Theorem 2.1 (Local Reciprocity Law)**.** *Suppose K is a non-archimedean local field. There exists a unique homomorphism*

$$\phi_K\colon K^* \to \mathit{Gal}(K^{ab}/K)$$

satisfying:

(a) For any uniformizer π of K, $\phi_K(\pi)$ is the Frobenius element of $\mathit{Gal}(K^{un}/K)$ under the restriction $\mathit{Gal}(K^{ab}/K) \to \mathit{Gal}(K^{un}/K)$.

(b) For any finite abelian extension L of K, there is an exact sequence:

$$1 \to \mathit{Nm}_{L/K}(L^*) \to K^* \to \mathit{Gal}(L/K) \to 1$$

where the latter map is the composition of ϕ_K and the restriction map. This induces an isomorphism

$$\phi_{L/K}\colon K^*/\mathit{Nm}_{L/K}(L^*) \to \mathit{Gal}(L/K)$$

In particular, $[K^* : \mathit{Nm}_{L/K}(L^*)] = [L : K]$.

*The map $\phi_{L/K}$ is then called the **local Artin map**.*

The following corollary can be deduced from Theorem 2.1.

Corollary 2.2. *Let K be a non-archimedean local field. Assume that Theorem 2.1 is true. Then*

(a) *The map $L \mapsto Nm(L^*)$ is as order-reversing bijection between abelian extensions of K and norm groups in K^*.*

(b) $Nm\big((L \cdot L')^*\big) = Nm(L^*) \cap Nm(L'^*)$.

(c) $Nm\big((L \cap L')^*\big) = Nm(L^*) \cdot Nm(L'^*)$

(d) *If a subgroup of K^* contains a norm group, then it is a norm group itself. Here the norm groups are $Nm(L^*)$ where L/K is an abelian finite extension.*

Proof. We prove in the order of (b)→(a)→(d)→(c).

(b) If $L \subset L'$, $\text{Nm}(L'^*) \subset \text{Nm}(L^*)$ since $\text{Nm}_{L'/K} = \text{Nm}_{L/K} \circ \text{Nm}_{L'/L}$. Thus,

$$\text{Nm}\big((L \cdot L')^*\big) \subset \text{Nm}(L^*) \cap \text{Nm}(L'^*)$$

Conversely, for any $a \in \text{Nm}(L^*) \cap \text{Nm}(L'^*)$, both $\phi_{L/K}(a), \phi_{L'/K}(a)$ are identities by Theorem 2.1. Since $\phi_{L \cdot L'/K}(a)|_L = \phi_{L/K}(a)$ and $\phi_{L \cdot L'/K}(a)|_{L'} = \phi_{L'}(a)$, $a \in \ker(\phi_{L \cdot L'/K}) = \text{Nm}(L \cdot L')$.

(a) We first show that the map in (a) is order-reversing. If $\text{Nm}(L^*) \supset \text{Nm}(L'^*)$, $\text{Nm}(L'^*) = \text{Nm}\big((L \cdot L')^*\big)$ by (b). Since

$$[L \cdot L' : K] = [K^* : \text{Nm}(L \cdot L')] = [K^* : \text{Nm}(L'^*)] = [L' : K]$$

we have $L \cdot L' = L'$. Thus, $L' \supset L$. Therefore, $L \mapsto \text{Nm}(L^*)$ is order-reversing. It follows that this map is injective. By definition, this map is surjective.

(d) Let $N = \text{Nm}(L^*)$ be a norm group and $N' \supset N$ is a subgroup of K^*. Let L' be the subfield of L fixed by $\phi_{L/K}(N'/N)$. Then N'/N is the kernel of the composition

$$K^*/N \overset{\phi_{L/K}}{\to} \text{Gal}(L/K) \to \text{Gal}(L'/K)$$

The composition is same to $\phi_{L'/K}$. Thus, $K^*/N' \cong \text{Gal}(L'/K)$ given by $\phi_{L'/K}$. Hence, $N' = \text{Nm}(L'^*)$.

(c) Note that $\text{Nm}(L^*) \cdot \text{Nm}(L'^*)$ is the smallest subgroup in K^* containing both $\text{Nm}(L^*)$ and $\text{Nm}(L'^*)$, and it is a norm group by (d). On the other hand, $L \cap L'$ is the biggest field contained in both L, L'. They must accord by (a).

□

Theorem 2.3 (Local Existence Theorem)**.** *The norm subgroups in K^* are equivalent to the open subgroups of finite index in K^*.*

The goal of this section is to prove Theorem 2.1 and Theorem 2.3.

The following remarks of the main theorems are essential to the proof. Recall in the finite case, if L/K is a totally ramified extension of degree n and F/K is an unramified extension of degree m, then LF/K is of degree mn (Here we do not require K, L, F to be local fields). Actually K^{ab} can also be decomposed into the composition of a maximal unramified extension and a maximal totally ramified extension as follows.

Given the isomorphisms

$$\phi_{L/K} \colon K^*/\mathrm{Nm}(L^*) \to \mathrm{Gal}(L/K) \cong \mathrm{Gal}(K^{ab}/K)/\mathrm{Gal}(K^{ab}/L)$$

for each finite abelian extension L of K, by passing to the limit we get an isomorphism:

$$\hat{\phi}_K \colon \widehat{K^*} \to \mathrm{Gal}(K^{ab}/K)$$

where $\widehat{K^*}$ is the profinite completion of K^* since $\mathrm{Nm}(L^*)$ are all open subgroups of finite index in K^* by Theorem 2.3.

Now choose an uniformizer π of K. We have

$$K^* \cong U_K \times \pi^{\mathbb{Z}} \cong U_K \times \mathbb{Z}$$

Lemma 2.4. *Under the decomposition above,* $\lim_{n\in\mathbb{N}^*, m\in\mathbb{N}^*} K^*/\big((1+\mathfrak{m}^n)\times m\mathbb{Z}\big) \cong \widehat{K^*}$.

Proof. It suffices to show that for any open subgroup of finite index H in K^*, H contains some $(1+\mathfrak{m}^n)\times m\mathbb{Z}$. Since H is open and $(1+\mathfrak{m}^n)\times\{0\}$ forms a fundamental system of neighborhoods of 1 in K^*, $H \supset (1+\mathfrak{m}^n)\times\{0\}$ for some n. Moreover, H contains a $u\pi^r$ for some integer r and $u \in U_K$. Since $U_K/(1+\mathfrak{m}^n)$ is a finite group, $u^s \in (1+\mathfrak{m}^n)$ for some integer s. Therefore, $H \supset (1+\mathfrak{m}^n)\times rs\mathbb{Z}$. □

Since U_K is profinite with respect to $1+\mathfrak{m}^n$, we have

$$\widehat{K^*} \cong U_K \times \pi^{\hat{\mathbb{Z}}} \cong U_K \times \hat{\mathbb{Z}}$$

It is well-known that profinite topological groups are equivalent to compact Hausdorff totally disconnected topological groups. Since $U_K, \hat{\mathbb{Z}}$ are profinite, they are compact. Because $\widehat{K^*}$ is Hausdorff, both $U_K, \hat{\mathbb{Z}}$ are closed subgroups in $\widehat{K^*}$. Since $\mathbb{Z}$ is dense in $\hat{\mathbb{Z}}$, $\hat{\mathbb{Z}} = \overline{\mathbb{Z}}$ in $\widehat{K^*}$. Let $K_\pi = (K^{ab})^{\hat{\phi}_K(\pi)}$ and $K^{un} = (K^{ab})^{\hat{\phi}_K(U_K)}$. Then by infinite Galois theory, $\mathrm{Gal}(K^{ab}/K_\pi) = \hat{\mathbb{Z}}$ and $\mathrm{Gal}(K^{ab}/K^{un}) = U_K$. Thus, K_π is the union of finite abelian extensions L such that $\pi \in \mathrm{Nm}(L^*)$, which are totally ramified, and K^{un} is the union of finite abelian extensions L such that $\mathrm{Nm}(L^*) \supset U_K$, which are unramified. We deduce that K^{un} is the maximal unramified extension of K in K^{ab} and $K^{un} \cap K_\pi = K$. Thus, $\mathrm{Gal}(K_\pi K^{un}/K) = \mathrm{Gal}(K_\pi/K) \times \mathrm{Gal}(K^{un}/K) = U_K \times \hat{\mathbb{Z}}$. Hence, $K^{ab} = K_\pi K^{un}$.

Under such view of point, we can show the uniqueness of ϕ_K.

Lemma 2.5. *Assume that Theorem 2.3 is true. Then there exists at most one homomorphism $\phi\colon K^* \to Gal(K^{ab}/K)$ satisfying the conditions in Theorem 2.1.*

Proof. We know that $K^{ab} = K^{un}K_\pi$. If there is a ϕ satisfies the conditions in Theorem 2.1, then $\phi(\pi)|_{K^{un}}$ is the Frobenius element for any uniformizer π of K. Since K_π is fixed by $\phi(\pi)$ from above discussion, the value of $\phi(\pi)$ is determined for all uniformizer π. Because K^* is generated by uniformizers π of $\mathcal{O}_K$, the value of ϕ is uniquely determined. □

Note that we know the restriction of the local Artin map on K^{un} is the Frobenius element. The proof of local class field theory consists of several steps:

(a) Constructing the fields K^{un}, K_π discussed above and the restriction of the local Artin map $U_K \to \mathrm{Gal}(K_\pi/K)$.

(b) Extend the map to $\phi_\pi\colon K^* \to \mathrm{Gal}(K_\pi K^{un}/K)$.

(c) Show that the composition $K_\pi K^{un}$ and the associated map ϕ_π are independent of the choice of π.

(d) Show that $K_\pi K^{un} = K^{ab}$.

(e) Show that ϕ_π satisfies the condition (b) of Theorem 2.1.

The construction of K^{un} will be displayed in the following of this subsection. The remaining parts are (a)(b)(c) are done in Subsection 2.3. Then (d) is proved in Subsection 2.4. Finally, (e) is shown in Subsection 2.5.

Example 2.6. *Suppose $K = \mathbb{Q}_p$ for some prime number p and pick the uniformizer $\pi = p$. By Kummer-Dedekind Theorem, for each positive integer n, $\mathbb{Q}_p(\mu_n)/\mathbb{Q}_p$ is unramified if $(n,p) = 1$ and is totally ramified if $n = p^i$ for some positive integer i. Moreover, the Galois group $Gal\big(\mathbb{Q}_p(\mu_n)/\mathbb{Q}_p\big)$ is $(\mathbb{Z}/n\mathbb{Z})^*$. By taking the colimit, we see that the Galois groups of $\left(\bigcup_{(n,p)=1} \mathbb{Q}_p(\mu_n)\right)/\mathbb{Q}_p$ and $\left(\bigcup_{i=1}^{\infty} \mathbb{Q}_p(\mu_{p^i})\right)/\mathbb{Q}_p$ are $\hat{\mathbb{Z}}$ and $(\mathbb{Z}_p)^*$ respectively. Thus, we have*

$$(\mathbb{Q}_p)_\pi = \bigcup_{i=1}^{\infty} \mathbb{Q}_p(\mu_{p^i}) \qquad Q_p^{un} = \left(\bigcup_{(n,p)=1} \mathbb{Q}_p(\mu_n) \right)$$

By above discussion,

$$\mathbb{Q}_p^{ab} = \left(\bigcup_{(n,p)=1} \mathbb{Q}_p(\mu_n) \right) \cdot \left(\bigcup_{i=1}^{\infty} \mathbb{Q}_p(\mu_{p^i}) \right)$$

The above method of construction Q_p^{un} applies to arbitrary local field K. Suppose $p \nmid n$, μ_n is the primitive n-th root of unity over K and $L = K(\mu_n)$. Suppose $\Phi_n(t)$ is the minimal polynomial of μ_n over K and $\overline{\Phi_n}(t)$ is the reduction of $\Phi_n(t)$ to the residue field k. Thus, $\overline{\Phi_n}(t) \mid (t^n - 1)$, so it is separable. By Hensel's Lemma, $\overline{\Phi_n}(t)$ is also irreducible. Thus, $\overline{\Phi_n}(t)$ is the minimal polynomial of $\bar{\mu_n}$ over k. Therefore,

$$[L : K] = \deg \Phi_n(t) = \deg \overline{\Phi_n}(t) = [k(\bar{\mu_n}) : k] \leqslant [l : k] \leqslant [L : K]$$

where l is the residue field of L. Hence, $[L : K] = [l : k]$ implying that L/K is unramified. By field theory, we know that $l = k(\bar{\mu_n})$ is the splitting field of $t^{q^f} - t$, where f is the smallest number such that $n \mid (q^f - 1)$. Therefore, $\left(\bigcup_{(n,p)=1} K(\mu_n)\right)/K$ is an unramified extension and has the residue field $\bar{k}$, implying that $K^{un} = \bigcup_{(n,p)=1} K(\mu_n)$.

However, we cannot simply add of roots of unity to K to construct K_π. Indeed, if $K = \mathbb{F}_p(\!(T)\!)$, then K itself contains p^i-th roots of unity. Lubin-Tate theory generalizes this method to arbitrary local field via Lubin-Tate formal group laws. If we let $\mathbb{G}_m$ to be the multiplication formal group law on $\mathbb{Z}_p$, i.e., $\mathbb{G}_m(X,Y) = X + Y + XY$, then there exists a natural map $\mathbb{Z} \to \mathrm{End}(\mathbb{G}_m)$ given by $n \mapsto \big((1+T)^n - 1\big)$. Then we see that $(\mu_{p^i} - 1)$ is a p^n-torsion point of $\mathbb{G}_m$. Thus, $\mathbb{Q}_p(\mu_{p^i}) = \mathbb{Q}_p(\mu_{p^i} - 1)$ can be viewed as adding p^n-torsion points in $\mathbb{Q}_p^{al}$.

2.2 Lubin-Tate Formal Group Laws

Note that for power series f, g, h, $f \circ (g+h) \neq f \circ g + f \circ h$ in general. In order to make the distribution law possible, we need to rewrite the addition. Suppose F is the new addition. Then we need $f \circ F(g,h) = F(f \circ g, f \circ h)$. We use the formal group law to capture this.

Definition 2.7 (One-Parameter Commutative Formal Group Law**)**. Let R be a commutative ring. A **(commutative one-parameter) formal group law** is a power series $F \in R[\![X,Y]\!]$ satisfying that

(a) $F(X,Y) \equiv X+Y \pmod{(X,Y)^2}$.

(b) (Associativity) $F\big(X, F(Y,Z)\big) = F\big(F(X,Y), Z\big)$.

(c) (Commutativity) $F(X,Y) = F(Y,X)$.

We can prove that with the conditions (a)(b), there exists a unique $i_F(T) \in R[\![T]\!]$ such that $F\big(X, i_F(X)\big) = 0$.

We denote End(F) by the set of $f \in R[\![T]\!]$ such that $f \circ F(X,Y) = F\big(f(X), f(Y)\big)$ and $f +_F g = F(f,g)$. Then we see from the beginning of this subsection that End(F) admits a ring structure with the addition $+_F$ and the multiplication $\circ$.

Definition 2.8. Let $\mathcal{F}_\pi$ be the set of $f(T) \in \mathcal{O}_K[\![T]\!]$ such that

(a) $f \equiv \pi T \pmod{T^2}$.

(b) $f \equiv T^q \pmod{\pi}$.

Example 2.9. *Let $K = \mathbb{Q}_p$, $\pi = p$. Then $f(T) = (1+T)^p - 1$ lies in $\mathcal{F}_p$.*

Lemma 2.10. *Suppose $f, g \in \mathcal{F}_\pi$ and $\phi_1(X_1, \cdots, X_n) \in \mathcal{O}_K[X_1, \cdots, X_n]$ is linear. Then there exists a unique $\phi \in \mathcal{O}_K[\![X_1, \cdots, X_n]\!]$ such that*

(a) $\phi \equiv \phi_1 \pmod{(X_1, \cdots, X_n)^2}$.

(b) $f\big(\phi(X_1, \cdots, X_n)\big) = \phi\big(g(X_1), \cdots, g(X_n)\big)$.

Proof. The idea is doing induction on the degree of ϕ and taking the limit, i.e., show that there exists a unique polynomial $\phi_r(X_1, \cdots, X_n)$ of degree r such that

$$\begin{cases} \phi_r \equiv \phi_1 \pmod{(X_1, \cdots, X_n)^2} \\ f\big(\phi_r(X_1, \cdots, X_n)\big) \equiv \phi_r\big(g(X_1), \cdots, g(X_n)\big) \pmod{(X_1, \cdots, X_n)^{r+1}} \end{cases}$$

When $r = 1$, this is just ϕ_1.

Suppose $r > 1$ and the above statement holds for $r-1$. Then we need to show that there is a unique homogeneous polynomial ψ_r of degree r such that $\phi_{r-1} + \psi_r$ satisfies

$$f \circ (\phi_{r-1} + \psi_r) \equiv (\phi_{r-1} + \psi_r) \circ g \pmod{(X_1, \cdots, X_n)^{r+1}}$$

Equivalently, we have

$$\begin{aligned} f \circ \phi_{r-1} + \pi \psi_r &\equiv \phi_{r-1} \circ g + \psi_r \circ \pi \pmod{(X_1, \cdots, X_n)^{r+1}} \\ f \circ \phi_{r-1} - \phi_{r-1} \circ g &\equiv (\pi^r - \pi)\psi_r \pmod{(X_1, \cdots, X_n)^{r+1}} \\ \psi_r &\equiv \frac{f \circ \phi_{r-1} - \phi_{r-1} \circ g}{\pi(\pi^{r-1} - 1)} \pmod{(X_1, \cdots, X_n)^{r+1}} \end{aligned}$$

The uniqueness is proved. Note that

$$f \circ \phi_{r-1} - \phi_{r-1} \circ g \equiv \phi_{r-1}^q(X_1, \cdots, X_n) - \phi_{r-1}(X_1^q, \cdots, X_n^q) \equiv 0 \pmod{\pi}$$

Thus, ψ_r is the degree r part of $\frac{f \circ \phi_{r-1} - \phi_{r-1} \circ g}{\pi(\pi^{r-1}-1)}$. Let $\phi = \phi_1 + \psi_2 + \psi_3 + \cdots$. Then ϕ satisfies condition (a). Note that for each r,

$$f \circ \phi \equiv f \circ \phi_r \equiv \phi_r \circ g \equiv \phi \circ g \pmod{(X_1, \cdots, X_n)^{r+1}}$$

Thus, $f \circ \phi = \phi \circ g$. $\square$

The following three propositions can be deduced by repeatedly applying the above lemma.

Proposition 2.11. *For every $f \in \mathcal{F}_\pi$, there is a unique formal group law $F_f \in \mathcal{O}_K[\![X, Y]\!]$ admitting f as an endomorphism.*

Proposition 2.12. *For $f, g \in \mathcal{F}_\pi$ and $a \in \mathcal{O}_K$, let $[a]_{g,f}$ be the unique element of $\mathcal{O}_K[\![T]\!]$ such that*

(a) $[a]_{g,f} \equiv aT \pmod{T^2}$.

(b) $g \circ [a]_{g,f} = [a]_{g,f} \circ f$.

Then $[a]_{g,f}$ is a homomorphism from F_f to F_g.

Proposition 2.13. *For any $a, b \in \mathcal{O}_K$, we have $[a+b]_{g,f} = [a]_{g,f} +_{F_g} [b]_{g,f}$ and $[ab]_{h,f} = [a]_{h,g} \circ [b]_{g,f}$.*

This proposition has two direct corollaries.

Corollary 2.14. *For any $f, g \in \mathcal{F}_\pi$, we have $F_f \cong F_g$.*

Proof. Given every $u \in \mathcal{O}_K^*$, $[u]_{f,g}$ and $[u^{-1}]_{g,f}$ are inverse to each other. □

Corollary 2.15. *For each $a \in \mathcal{O}_K$, there is a unique endomorphism $[a]_f \colon F_f \to F_f$ such that $[a]_f \equiv aT \pmod{T^2}$. The map*

$$\mathcal{O}_K \to \mathit{End}(F_f)\colon a \mapsto [a]_f$$

is a ring isomorphism. In particular, we have $[\pi]_f = f$.

The formal group law F_f associated to an uniformizer π is called the **Lubin-Tate formal group law**.

Example 2.16. *When $K = \mathbb{Q}_p$, $\pi = p$, $f(T) = (1+T)^p - 1$, $F_f = \mathbb{G}_m = X + Y + XY$ is the multiplicative formal group law. When $a \in \mathbb{Z}$, the power series $[a]_f = (1+T)^a - 1$. This can be extended to $\mathbb{Z}_p$. For any $a \in \mathbb{Z}_p$,*

$$(1+T)^a := \sum_{m \geqslant 0} \binom{a}{m} T^m \qquad \binom{a}{m} := \frac{a(a-1)\cdots(a-m+1)}{m(m-1)\cdots 1}$$

By continuity, $\binom{a}{m} \in \mathbb{Z}_p$ and $[a]_f := \big((1+T)^a - 1\big) \in \mathit{End}(\mathbb{G}_m)$.

Example 2.17. *When $K = \mathbb{F}_p((Z))$, the general situation is complicated. A simple example is the case of $p = 2$. $f(T) = ZT + T^2 \in \mathcal{F}_\pi$. Then $F_f = \mathbb{G}_a = X + Y$ is the additive formal group law and $[a]_f = \sum_{i=0}^{\infty} a_i T^{2^i}$, where $a_0 = a$ and $a_i = \frac{a_{i-1}^2 - a_{i-1}}{Z(Z^{2^i - 1} - 1)}$ for $i > 1$. The formula is obtained by going through the proof of Lemma 2.10.*

2.3 Construction of K_π and the Local Artin Map

For any $f \in \mathcal{F}_\pi$, let $\Lambda_f = \{\alpha \in K^{al} : |\alpha| < 1\}$. Define a $\mathcal{O}_K$-module structure on Λ_f by $\alpha + \beta := \alpha +_{F_f} \beta$ and $a \cdot \alpha := [a]_f(\alpha)$. Let $\Lambda_{f,n}$ be the submodule of Λ_f consisting of elements killed by $[\pi]_f^n$.

Remark. *The canonical isomorphism* $[1]_{g,f}\colon F_f \to F_g$ *induces isomorphisms* $\Lambda_f \to \Lambda_g$ *and* $\Lambda_{f,n} \to \Lambda_{g,n}$ *for each* n.

Proposition 2.18. *For each* n*, we have that* $\Lambda_{f,n} \cong \mathcal{O}_K/(\pi^n)$ *as* $\mathcal{O}_K$*-modules. Therefore, End*$(\Lambda_{f,n}) \cong \mathcal{O}_K/(\pi^n)$ *and Aut*$(\Lambda_{f,n}) \cong \big(\mathcal{O}_K/(\pi^n)\big)^*$.

Proof. By the above remark, it suffices to take $f = \pi T + T^q$. Thus, $[\pi^n]_f = \pi^n T + \cdots + T^{q^n}$. From the Newton polygon of $[\pi^n]_f$, we see that all the roots of $[\pi^n]_f$ lie in $\Lambda_{f,n}$.

Since $f = \pi T + T^q$ is an Eisenstein polynomial, f is irreducible and has q distinct roots. Thus, $\Lambda_{f,1}$ has exactly q elements. By the structure theorem of modules over PID, $\Lambda_{f,1} \cong \mathcal{O}_K/(\pi)$ since $\mathcal{O}_K/(\pi^n)$ contains q^n elements.

For each $\alpha \in K^{al}$ with $|\alpha| < 1$, $f(T) - \alpha = T^q + \cdots + \pi T - \alpha$. From the Newton polygon of $f(T) - \alpha$, we see that all roots of $f(T) - \alpha$ lie in Λ_f. Therefore, $[\pi]_f$ is surjective.

Suppose $\Lambda_{f,n} \cong \mathcal{O}_K/(\pi^n)$ for some n. Since $[\pi]_f$ is surjective, we have the following exact sequence:

$$0 \to \Lambda_{f,1} \to \Lambda_{f,n+1} \overset{[\pi]_f}{\to} \Lambda_{f,n} \to 0$$

Thus, $\Lambda_{f,n+1}$ has q^{n+1} elements. Suppose $\Lambda_{f,n+1} \cong \mathcal{O}_K/(\pi^{n_1}) \oplus \cdots \mathcal{O}_K/(\pi^{n_r})$ by the structure theorem of modules over PID. Then the exact sequence implies that $\Lambda_{f,1} \cong (\pi^{n_1-1})/(\pi^{n_1}) \oplus \cdots \oplus (\pi^{n_r-1})/(\pi^{n_r})$. Therefore, $r = 1$ and $\Lambda_{f,n+1} \cong \mathcal{O}_K/(\pi^{n+1})$. □

Lemma 2.19. *Every subfield* E *in* K^{al} *containing* K *is closed in the topological sense.*

Proof. Let $G = \mathrm{Gal}(K^{al}/E)$. By the uniqueness of the extension of the absolute valuation, $\|\tau(\cdot)\| = \|\cdot\|$ for any $\tau \in G$. Suppose $x \in \overline{E}$ is a limit of $x_n \in E$. Then

$$\|\tau(x) - x_n\| = \|\tau(x - x_n)\|$$

also converge to zero, so $\tau(x) \in \overline{E}$. Therefore, $\overline{E} = (K^{al})^G = E$. □

Theorem 2.20. *Let* $K_{\pi,n} = K(\Lambda_{f,n})$*. Then we have*

(a) $K_{\pi,n}$ *is independent of the choice of* f.

(b) For each n*,* $K_{\pi,n}/K$ *is a totally ramified extension of degree* $(q-1)q^{n-1}$.

(c) The action of $\mathcal{O}_K$ *on* Λ_n *induces an isomorphism*

$$(\mathcal{O}_K/\mathfrak{m}^n)^* \to \mathit{Gal}(K_{\pi,n}/K)$$

Thus, $K_{\pi,n}/K$ is an abelian extension.

(d) *For each n, we have $\pi \in Nm(K^*_{\pi,n})$.*

Proof. (a) Suppose $g \in \mathcal{F}_\pi$. Via the isomorphism $[1]_{g,f} \colon \Lambda_{f,n} \to \Lambda_{g,n}$, we have that

$$\widehat{K(\Lambda_{g,n})} = \widehat{K\big([1]_{g,f}(\Lambda_{f,n})\big)} \subset \widehat{K\big(\Lambda_{f,n}\big)} = \widehat{K\big([1]_{f,g}(\Lambda_{g,n})\big)} \subset \widehat{K(\Lambda_{g,n})}$$

Thus, $\widehat{K(\Lambda_{g,n})} = \widehat{K(\Lambda_{f,n})}$. By the above lemma,

$$K(\Lambda_{g,n}) = \widehat{K(\Lambda_{g,n})} \cap K^{al} = \widehat{K(\Lambda_{f,n})} \cap K^{al} = K(\Lambda_{f,n})$$

(b)(c) Since $K_{\pi,n}$ is independent on the choice of f, we may assume again that $f = [\pi]_f = \pi T + \cdots + T^q$.

Choose a nonzero root π_1 of f and π_{s+1} of $f(X) - \pi_s$ for each $s = 1, 2, \cdots, n-1$. Then there is a sequence of field extensions:

$$K(\pi_n) \supset K(\pi_{n-1}) \supset \cdots \supset K(\pi_1) \supset K$$

Note that each extension is Eisenstein, so each $K(\pi_n)/K$ is totally ramified. The degree of $K(\pi_1)/K$ is $q-1$ and the degree of $K(\pi_{s+1})/K(\pi_s)$ is q for each s. Therefore, $K(\pi_n)/K$ is a totally ramified extension of degree $q^{n-1}(q-1)$. Since $[\pi^n]_f(\pi_n) = 0$, $K(\Lambda_{f,n}) \supset K(\pi_n)$.

Since $K(\Lambda_{f,n})$ is the splitting field of $[\pi^n]_f$ over K, Gal$\big(K(\Lambda_{f,n})/K\big)$ is isomorphic to a subgroup of permutations on $\Lambda_{f,n}$. It is easy to show the action of Gal$\big(K(\Lambda_{f,n})/K\big)$ on $\Lambda_{f,n}$ is compatible with the A-module structure on $\Lambda_{f,n}$. Thus, Gal$\big(K(\Lambda_{f,n})/K\big) <$ Aut$(\Lambda_{f,n}) = \big(\mathcal{O}_K/(\pi^n)\big)^*$. Therefore,

$$(q-1)q^{n-1} = |\big(\mathcal{O}_K/(\pi^n)\big)^*| \geqslant [K(\Lambda_{f,n})/K] \geqslant [K(\pi_n)/K] = (q-1)q^{n-1}$$

Hence, $K(\Lambda_{f,n}) = K(\pi_n)$ is a totally ramified extension of degree $(q-1)q^{n-1}$ over K and Gal$(K_{\pi,n}/K) \cong (\mathcal{O}_K/\mathfrak{m}^n)^*$ and $u \in \mathcal{O}_K^*$ acts on $\Lambda_{f,n}$ by $[u]_f$.

(d) Since the degree of $[\pi^n]_f(T)/T = \pi + \cdots + T^{(q-1)q^{n-1}}$ is $(q-1)q^n$, it is the minimal polynomial of π_n over K. Hence, $\mathrm{Nm}_{K_{\pi,n}/K}(\pi_n) = (-1)^{(q-1)q^{n-1}}\pi$, so $\pi \in \mathrm{Nm}(K^*_{\pi,n})$.

□

Let $K_\pi = \cup_{n=1}^{\infty} K_{\pi,n}$. By passing to the limit, we have that $\tilde{\phi}_f\colon U_K \cong \mathrm{Gal}(K_\pi/K)$ given by $u \mapsto [u^{-1}]_f$. The inverse here will make the formula elegant in the future.

Let $\phi_f\colon K^* \to \mathrm{Gal}(K_\pi K^{un}/K)$ given as follows: for each $a = u\pi^m \in K^*$, $\phi_f(a)|_{K^{un}}$ is the m-th power of the Frobenius element and $\phi_f(a)(\lambda) = \tilde{\phi}_f(u)(\lambda) = [u^{-1}]_f(\lambda)$ for all $\lambda \in \bigcup_{n=1}^{\infty} \Lambda_{f,n}$.

Next, we want to show that $K_\pi K^{un}$ and ϕ_f are independent of the choice of π, f. Note that in the proof of the part (a) of Theorem 2.20, the essential part is the $\mathcal{O}_K$-isomorphism $[1]_{g,f}\colon \Lambda_{f,n} \to \Lambda_{g,n}$, where $[1]_{g,f}$ is a power series with coefficients in $\mathcal{O}_K$. We also want such an isomorphism for different uniformizers. Now suppose π, ω are two uniformizers of $\mathcal{O}_K$ and $\omega = u\pi$ for some $u \in U_K$. Let $B, \hat{B}$ be the integer ring of $K^{un}, \hat{K}^{un}$ respectively. Suppose we have such a $\mathcal{O}_K$-isomorphism $\theta\colon \Lambda_{f,n} \to \Lambda_{g,n}$, where $f \in \mathcal{F}_\pi$, $g \in \mathcal{F}_\omega$ and θ is a power series with coefficients in $\hat{B}$ (Since we took completion in the proof of the part (a) of Theorem 2.20, the coefficients of θ to can be taken in $\hat{B}$ and the proof of part (a) of Theorem 2.20 still work). We need to explore properties θ needed for proving that ϕ_f is independent of π, f.

In order to show that $\phi_f = \phi_g$, it suffices to show that they agree on every uniformizer of $\mathcal{O}_K$. Given any uniformizer π' of $\mathcal{O}_K$, $\phi_f(\pi')|_{K^{un}} = \phi_g(\pi')|_{K^{un}}$ is the Frobenius element. Suppose $\pi' = v\pi = vu^{-1}\omega$. Let θ^σ be the power series obtained by acting σ on each coefficient of θ. Then for each $\lambda \in \Lambda_{f,n}$,

$$\phi_f(\pi')\big(\theta(\lambda)\big) = \theta^\sigma\big(\phi_f(v)(\lambda)\big) = \theta^\sigma \circ [v^{-1}]_f(\lambda)$$

We expect that the right-hand side is equal to $\phi_g(\pi')\big(\theta(\lambda)\big) = [uv^{-1}]_g \circ \theta(\lambda) = \theta \circ [uv^{-1}]_f(\lambda)$ since θ is a $\mathcal{O}_K$-homomorphism. Therefore, we need that $\theta^\sigma = \theta \circ [u]_f$. This implies that θ induces isomorphisms $\Lambda_{f,n} \to \Lambda_{g,n}$ because $(\sigma \circ f)^\sigma = \theta \circ [u\pi]_f = [\omega]_g \circ \theta = g \circ \theta$.

Suppose $\theta(T) = \epsilon T + \cdots$ for some $\epsilon \in \hat{B}$. Then $\sigma(\epsilon) = \epsilon u$. We claim that $\sigma(\cdot)/\cdot\colon \hat{B} \to \hat{B}$ is surjective while it is not true that $\sigma(\cdot)/\cdot\colon B \to B$ is surjective. That is why we require the coefficients of θ to be in $\hat{B}$.

Lemma 2.21. *The homomorphism $\sigma(\cdot)/\cdot\colon \hat{B}^* \to \hat{B}^*$ is surjective with kernel $\mathcal{O}_K^*$.*

Proof. Let $\mathfrak{n}$ be the maximal ideal in B. It suffices to show that the sequence

$$1 \to (\mathcal{O}_K/\mathfrak{m}^n)^* \to (B/\mathfrak{n}^n)^* \overset{\sigma(\cdot)/\cdot}{\to} (B/\mathfrak{n}^n)^* \to 1$$

is exact for each n and then pass to the limit.

For $n = 1$, $B/\mathfrak{n} = k^{al}$ and the result follows easily. Assume that the sequence is exact for $n-1$. Then we have the following diagram:

$$\begin{array}{ccccccccc}
 & & 1 & & & & 1 & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & (\mathcal{O}_K/\mathfrak{m})^* & & & & (\mathcal{O}_K/\mathfrak{m}^{n-1})^* & & \\
 & & \downarrow & & & & \downarrow & & \\
1 & \longrightarrow & (B/\mathfrak{n})^* & \longrightarrow & (B/\mathfrak{n}^n)^* & \longrightarrow & (B/\mathfrak{n}^{n-1})^* & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \sigma(\cdot)/\cdot} & & \downarrow{\scriptstyle \sigma(\cdot)/\cdot} & & \downarrow{\scriptstyle \sigma(\cdot)/\cdot} & & \\
1 & \longrightarrow & (B/\mathfrak{n})^* & \longrightarrow & (B/\mathfrak{n}^n)^* & \longrightarrow & (B/\mathfrak{n}^{n-1})^* & \longrightarrow & 1 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 1 & & & & 1 & &
\end{array}$$

By the snake lemma, $\sigma(\cdot)/\cdot\colon (B/\mathfrak{n}^n)^* \to (B/\mathfrak{n}^n)^*$ is surjective with kernel of q^n elements. Since $(\mathcal{O}_K/\mathfrak{m}^n)^*$ contains q^n elements and is contained in the kernel, the kernel is $(\mathcal{O}_K/\mathfrak{m}^n)^*$. □

The following proposition says that there exists the required $\theta \in \hat{B}[\![T]\!]$, so it finishes the proof that $K_\pi K^{un}$ and ϕ_f are independent on the choice of π, f.

Proposition 2.22. *Let $f \in \mathcal{F}_\pi$ and $g \in \mathcal{F}_\omega$, where $\omega = u\pi$ are two uniformizers of $\mathcal{O}_K$. Then there exists an $\epsilon \in \hat{B}^*$ such that $\sigma(\epsilon) = \epsilon u$ and a power series $\theta \in \hat{B}[\![T]\!]$ such that*

(a) $\theta(T) \equiv \epsilon T \pmod{T^2}$.

(b) $\theta^\sigma = \theta \circ [u]_f$.

(c) $\theta\big(F_f(X,Y)\big) = F_g\big(\theta(X), \theta(Y)\big)$.

(d) $\theta \circ [a]_f = [a]_g \circ \theta$.

Proof. The proof has four steps:

1. Show that there exists a $\theta \in \hat{B}[\![T]\!]$ satisfying (a)(b). This can be shown by induction on the degree of θ as Lemma 2.10.

2. Show that the θ in the first step can be chosen so that $g = \theta^\sigma \circ f \circ \theta^{-1}$. Let $h = \theta^\sigma \circ f \circ \theta^{-1}$. Then show that $h \in \mathcal{O}_K[\![T]\!]$. Let $\theta' = [1]_{g,h} \circ \theta$. Then θ' satisfies (a)(b) and $(\theta')^\sigma \circ f \circ (\theta')^{-1} = [1]_{g,h} \circ h \circ [1]_{h,g} = g$.

3. Show that $\theta\Big(F_f\big(\theta^{-1}(X),\theta^{-1}(Y)\big)\Big) = F_g(X,Y)$.

4. Show that $\theta\circ[a]_f\circ\theta^{-1} = [a]_g$.

Both the third and the fourth steps can be shown by directly applying Lemma 2.10. For details, see [Mil20] Proposition 3.10. □

2.4 Local Kronecker-Weber Theorem

The main propose of this section is to prove the following theorem:

Theorem 2.23. *(Local Kronecker-Weber Theorem)* $K_\pi K^{un} = K^{ab}$.

Lemma 2.24. *Suppose L is an abelian extension of K_π of degree m. Let K_m be the unique unramified extension of K_π of degree m. Then there exists an abelian totally ramified subextension L_t/K_π of L/K_π such that $L\subset L_tK_m = LK_m$.*

Proof. Note that $\mathrm{Gal}(LK_m/K_\pi)$ is a subgroup of $\mathrm{Gal}(L/K_\pi)\times\mathrm{Gal}(K_m/K_\pi)$, so every element in $\mathrm{Gal}(LK_m/K_\pi)$ has torsion m. Pick a $\tau\in\mathrm{Gal}(LK_m/K_\pi)$ such that $\tau|_{K_m}$ is the Frobenius element. Then τ has order m in $\mathrm{Gal}(LK_m/K_\pi)$. By the structure theorem of finite abelian groups, we have that $\mathrm{Gal}(LK_m/K_\pi)$ can be decomposed into $\langle\tau\rangle\times H$ for some subgroup $H<\mathrm{Gal}(LK_m/K_\pi)$. Let $L_t = L^{\langle\tau\rangle}$. Then $L_t\cap K_m = K_\pi$ since $\mathrm{Gal}(K_m/K_\pi) = \langle\tau|_{K_m}\rangle$, so L_t/K_π is totally ramified and $\mathrm{Gal}(L_t/K_\pi) = H$. Therefore, $L_tK_m = LK_m\supset L$. □

Remark. *The above proof actually works for all henselian valuation field with finite residue field K and finite abelian extension L/K.*

Lemma 2.25. *Any abelian totally ramified extension of K_π equals K_π.*

Proof. See [Mil20] Lemma 4.9.

Suppose L/K_π is an abelian totally ramified extension. The idea is that $\mathrm{Gal}(L/K_\pi) = \bigcap_{n=1}^{\infty}\mathrm{Gal}(L/K_{n,\pi}) = 1$. In fact, $\mathrm{Gal}(L/K_{\pi,n})$ is some ramification group of $\mathrm{Gal}(L/K)$, so their intersection is trivial. □

Lemma 2.26. *Suppose L is a finite unramified extension of K_π. Then $L\subset K_\pi K^{un}$.*

Proof. We have $L = K_\pi(\alpha)$ for some $\alpha\in K^{al}$. Suppose $f\in\mathcal{O}_{K_\pi}[T]$ is the minimal polynomial of α over K_π. Then $f\in\mathcal{O}_{K_{\pi,n}}[T]$ for some n. Since L/K_π is henselian, f is irreducible in the residue field of K_π, which is the same with the residue field of $K_{\pi,n}$. Thus, $K_{\pi,n}(\alpha)/K_{\pi,n}$ is unramified. Suppose U/K is the maximal unramified subextension of $K_{\pi,n}(\alpha)/K$, so the residue field of U equals the residue field of $K_{\pi,n}(\alpha)$. Then $[U : K]$ equals the inertia index of $K_{\pi,n}(\alpha)/K$, so $[U : K] = [K_{\pi,n}(\alpha) : K_{\pi,n}]$. Thus, $K_{\pi,n}(\alpha) = UK_{\pi,n}$. Hence, $L = K_\pi U \subset K_\pi K^{un}$. □

Proof. (of Theorem 2.23): Suppose L/K is a finite abelian extension. Then LK_π/K_π is also a finite abelian extension. Thus, there exists a totally ramified extension L_t/K_π and an unramified extension K_m/K_π such that $LK_\pi \subset L_t K_m$. By the two lemmas above, $L_t = K_\pi$ and $K_m \subset K_\pi K^{un}$. Therefore, $L \subset LK_\pi \subset K_\pi K^{un}$. Hence, $K_\pi K^{un} = K^{ab}$. □

2.5 Finishing of the Proof

Now we finish the proof of the main theorems of local class field theory by showing that the ϕ_K we constructed satisfies the Theorem 2.1 and that Theorem 2.3 is true.

By construction, we know that $\phi_K(\pi)|_{K^{un}}$ is the Frobenius element for each uniformizer π of K.

To prove the part (b) of the Theorem 2.1, take a finite abelian extension L/K.

Lemma 2.27. *The following diagram is commutative*

$$\begin{array}{ccc} L^* & \xrightarrow{\phi_L} & Gal(K^{ab}/L) \\ {\scriptstyle Nm}\downarrow & & \downarrow \\ K^* & \xrightarrow[\phi_K]{} & Gal(K^{ab}/K) \end{array}$$

Proof. Since L^* is generated by all uniformizers, it suffices to show that $\phi_L(\Pi) = \phi_K\big(\mathrm{Nm}(\Pi)\big)$ for all uniformizers Π of L. By taking the maximal unramified extension of K in L, it suffices to show the cases when L/K is totally ramified and unramified respectively.

For details, see [Iwa86] Theorem 6.9. □

Thus, ϕ_K induces a homomorphism $\phi_{L/K}\colon K^*/\mathrm{Nm}(L^*) \to \mathrm{Gal}(L/K)$.

From the construction of ϕ_K, it is easy to see that

Lemma 2.28. *The homomorphism ϕ_K is injective and continuous. Moreover, $\phi_K(K^*)$ is dense in $Gal(K^{ab}/K)$, consisting of all elements τ such that $\tau|_{K^{un}}$ is a power of the Frobenius element.*

The following proposition finishes the proof of the part (b) of Theorem 2.1.

Proposition 2.29. *As notations above, $\phi_{L/K}\colon K^*/Nm(L^*) \to Gal(L/K)$ is an isomorphism.*

Proof. Suppose $\phi_K(x)|_L = Id$ for some $x \in K^*$. Let $U = L \cap K^{un}$. Suppose $[U : K] = m$. Then $\phi_K(x)|_U = Id$ implies that $\phi_K(x)|_{K^{un}}$ is a power of σ^m by the above lemma. Note that $\mathrm{Gal}(K^{un}/U) \cong \mathrm{Gal}(LK^{un}/L) = \mathrm{Gal}(L^{un}/L)$ and σ^m corresponds to the Frobenius element of L under this isomorphism. Therefore, $\phi_K(x)|_{L^{un}}$ is a power of the Frobenius element of L^{un}/L. By the above lemma again, there is $y \in L$ such that $\phi_L(y) = \phi_K(x)$. Since $\phi_L(y) = \phi_K\big(\mathrm{Nm}(y)\big)$ and ϕ_K is injective, $x = \mathrm{Nm}(y)$. Thus, $\phi_{L/K}$ is injective.

In order to prove the surjectivity, identify $\mathrm{Gal}(L/K)$ as $\mathrm{Gal}(K^{ab}/K)/\mathrm{Gal}(K^{ab}/L)$. For each $[\tau] \in \mathrm{Gal}(L/K)$, $\tau\mathrm{Gal}(K^{ab}/L)$ is an open subset of $\mathrm{Gal}(K^{ab}/K)$. Since $\phi_K(K^*)$ is dense in $\mathrm{Gal}(K^{ab}/K)$, there is $x \in K^*$ such that $\phi_K(x) \in \tau\mathrm{Gal}(K^{ab}/L)$. Therefore, $\phi_{L/K}(x) = [\tau]$. □

Finally, we should prove Theorem 2.3.

Lemma 2.30. *Let K be a non-archimedean local field and L/K is a field extension. If $[K : Nm(L^*)]$ is finite, then $Nm(L^*)$ is open.*

Proof. Since U_L is profinite, U_L is compact. Thus, $\mathrm{Nm}(U_L)$ is compact in K^*, which is Hausdorff. Therefore, $\mathrm{Nm}(U_L)$ is closed in K^*. Since $\mathrm{Nm}(U_L) = \mathrm{Nm}(L^*) \cap U_K$, U_L is a closed subgroup with finite index in U_K, so is open in U_K. Since U_K is open in K^*, U_L is also open in K^*. Thus, $\mathrm{Nm}(L^*) \supset U_L$ is open. □

Proof. (of Theorem 2.3): By the part (b) of Theorem 2.1, we see that every norm group in K^* is of finite index. Thus, by the lemma above, they are open. Conversely, by the part (d) of the Corollary 2.2, it suffices to show that each open subgroup of finite index H in K^* contains a norm group. Since H is open, $H \supset (1 + \mathfrak{m}^n)$ for some n. Since H is of finite index, there is an integer s such that $H \supset (1 + \mathfrak{m}^n) \times s\mathbb{Z}$ by the same proof as in Lemma 2.4. Let K_s be the unramified extension of K of degree s and $L = K_{\pi,n}K_s$. Therefore, $\phi_{L/K}\big((1+\mathfrak{m}^n) \times s\mathbb{Z}\big) = 1$. It follows that $(1+\mathfrak{m}^n) \times s\mathbb{Z} \subset \mathrm{Nm}(L^*)$. Since they have the same index in K^*, $(1+\mathfrak{m}^n) \times s\mathbb{Z} = \mathrm{Nm}(L^*)$. □

3 Background in Algebraic Topology for Ando's Theorem on Norm-Coherent Coordinates

In this section we introduce some backgrounds in algebraic topology. We will omit most details, intending to provide an intuitive and quick introduction to Ando's theorem on norm-coherent coordinates. All topological spaces below are assumed to be pointed.

3.1 Generalized Cohomology and Homology Theories and Spectra

It is well-known that the singular cohomology and homology theory are characterized by several axioms on the functors, called the Eilenberg-Steenrod axioms. Actually there are other cohomology and homology theories share similar properties. We can generalize such axioms by dropping out the dimension axiom. It turns out that the resulted generalized cohomology and homology theories are very useful.

Definition 3.1 (Generalized Cohomology and Homology Theory)**.** A **generalized cohomology theory** is a sequence of contravariant functors h^n from the homotopy category of pointed CW-complexes to abelian groups satisfying the excision axiom with isomorphisms $\partial^n \colon h^{n+1} \circ \Sigma \to h^n$ such that for each cofiber sequence $A \xrightarrow{i} X \xrightarrow{j} X/A \xrightarrow{q} \Sigma A$, there is a long exact sequence

$$\cdots \xrightarrow{i^*} h^{n-1}(A) \xrightarrow{\delta} h^n(X/A) \xrightarrow{j^*} h^n(X) \xrightarrow{i^*} h^n(A) \xrightarrow{\delta} \cdots$$

where δ is the composition of q^* and ∂^n. Moreover, the sequence is natural.

A **generalized homology theory** is just the dual definition.

Actually such algebraic objects can be constructed from some geometric objects.

Definition 3.2 (Spectrum)**.** (a) A **prespectrum** E is a family of pointed topological spaces $\{E_n\}_{n\in\mathbb{Z}}$ and the structure maps $\Sigma E_n \to E_{n+1}$, where ΣE_n is the suspension of E_n.

(b) A **spectrum** is a prespectrum E such that the adjoint maps of the structure maps $E_n \to \Omega E_{n+1}$ (we will also call these the structure maps) are weak equivalences, where ΩE_{n+1} is the loop space of E_{n+1}.

(c) For a spectrum E, the **homotopy groups of** E is well-defined by

$$\pi_n(E) := \pi_{n+k}(E_k), n + k \geqslant 0$$

(d) Suppose E, F are two spectra. A map $f\colon E \to F$ between spectra is a sequence of maps $f_n\colon E_n \to F_n$ such that the following diagram commutes for each n

$$\begin{array}{ccc} E_n & \xrightarrow{f_n} & F_n \\ \downarrow & & \downarrow \\ \Omega E_{n+1} & \xrightarrow[\Omega f_{n+1}]{} & \Omega F_{n+1} \end{array}$$

(e) Suppose E is a spectrum. Then $\Sigma^m E$ is the spectrum defined by $(\Sigma^m E)_n := E_{m+n}$.

(f) Let f, g be two maps between spectra E, F. Then f, g are said to be **homotopic** if there is a map $H\colon I \to Sp(E, F)$ such that $H(0) = f$ and $H(1) = g$, where $Sp(E, F)$ is the set of morphisms between E, F. This is same to say a morphism $H'\colon E \to F^I$, where $F^I_n = \mathrm{Hom}(I, F_n)$ is a spectrum [Rez98].

Example 3.3. *Given a space X, we can define the $\Sigma^\infty X'$ by $(\Sigma^\infty X')_n := \Sigma^n X$ if $n \geqslant 0$ and just a point if $n < 0$. This is surely a prespectrum. However, it is not a spectrum. The structure maps are just injective. We can a make it to a spectrum by a process called* ***spectrification****. If there is a spectrum E_n with injective structure maps $\omega_n\colon E_n \to \Omega E_{n+1}$, then we define $(\mathbb{L}E)_n := colim_k \Omega^k E_{n+k}$ and $(\mathbb{L}\omega)_n := colim_k \Omega^k \omega_{n+k}$. It can be shown that the result sequence of spaces with structure maps is a spectrum and the spectrification is left adjoint to the natural inclusion functor from spectra to prespectra [EKMM97]. From the construction, we see that the homotopy groups invariant after the spectrification. We define the $\Sigma^\infty X$ to be the spectrification of $\Sigma^\infty X'$. In particular, we define the* ***sphere spectrum*** *S as the suspension spectrum of S^0.*

It can be shown that Σ^∞ is left adjoint to the functor from spectra to spaces by taking the space at degree 0 [Lur17, Section 1.4]. Therefore, maps between $\Sigma^\infty X$ and E is the same with pointed maps between X and E_0. Similarly, $[\Sigma^\infty X, E] = [X, E_0]$.

We can further define the smash product between spectra. However, the precise definition is very tedious. (See [EKMM97] for example) We just point out here the smash product makes the homotopy category of spectra into a monoidal category with the unit element S.

Definition 3.4. A **ring spectrum** is a spectrum with the unit map $\eta\colon S \to E$ and the multiplication map $m\colon E \wedge E \to E$, such that the following diagrams commute up to homotopy

$$\begin{array}{ccc} E & \xrightarrow{\eta \wedge Id_E} & E \wedge E \\ {\scriptstyle Id_E \wedge \eta}\downarrow & \searrow^{Id_E} & \downarrow{\scriptstyle m} \\ E \wedge E & \xrightarrow[m]{} & E \end{array}$$

$$\begin{array}{ccc} E \wedge E \wedge E & \xrightarrow{m \wedge Id_E} & E \wedge E \\ {\scriptstyle Id_E \wedge m}\downarrow & & \downarrow{\scriptstyle m} \\ E \wedge E & \xrightarrow[m]{} & E \end{array}$$

Definition 3.5. Let E be a spectrum. The **generalized cohomology and homology theory associated with** E, E^* and E_*, are defined by

$$E^n(X) := [\Sigma^{-n}X, E]$$
$$E_n(X) := \pi_n(X \wedge E)$$

for any spectrum X. This is a generalized cohomology theory by [Ada95, Chapter III, Proposition 6.1]

If E is a ring spectrum, we define the **coefficient ring** of E as the ring $E^{-*}(S) = \pi_*(E) = E_*(S)$. The ring structure of the coefficient ring is induced by the ring structure on E. We will simply denote it as E_*.

Example 3.6. *(a) Let $K(A, n)$ be the Eilenberg-Maclane space. Then $\Omega K(A, n + 1) \simeq K(A, n)$. Define the Eilenberg-Maclane spectrum HA by the spectrification of $HA'_n := K(A, n)$ for $n \geqslant 0$ and a point for $n < 0$. Then $HA_n = K(A, n)$ for $n \geqslant 0$, $HA^n(X) = H^n(X; A)$ and $HA_n(X) = H_n(X; A)$.*

(b) For the sphere spectrum S and a pointed space X,

$$S_n(X) = \pi_n(\Sigma^\infty X \wedge S) = \pi_n(\Sigma^\infty X) = \pi_n^S(X)$$

is the degree n stable homotopy group of X.

(c) Suppose X is a pointed space and E is a spectrum. Then

$$\begin{aligned} E^n(\Sigma^\infty X) &:= [\Sigma^{-n}\Sigma^\infty X, E] \\ &= [\Sigma^\infty X, \Sigma^n E] \\ &= [X, E_n] \end{aligned}$$

Besides the axioms given in the definition of generalized cohomology theories, the generalized cohomology theories associated with spectra have another important property, which is sometimes called the **additivity axiom** or the **wedge axiom**.

Proposition 3.7. *Suppose E is a spectrum. Then*

$$E^*(\vee_{\alpha\in I}X_\alpha) \cong \prod_{\alpha\in I} E^*(X_\alpha)$$

Proof. By definition,

$$E^n(\vee_{\alpha\in I}X_\alpha) = [\vee_{\alpha\in I}X_\alpha, E_n] \cong \prod_{\alpha\in I}[X_\alpha, E_n] = \prod_{\alpha\in I} E^*(X_\alpha)$$

□

A beautiful and fundamental result is that there is a correspondence between spectra and generalized cohomology theories with the wedge axiom.

Theorem 3.8 (Brown Representability Theorem)**.** *If h^* is a generalized cohomology theory satisfying*

$$h^*(\vee_{\alpha\in I}X_\alpha) \cong \prod_{\alpha\in I} h^*(X_\alpha)$$

then there is a spectrum E, such that $h^* = E^*$. If E is a ring spectrum, the associated generalized cohomology theory is called ***multiplicative****.*

Proof. For further references, see [Ada95, Chapter III, Remark 6.5]. □

3.2 Complex Orientations

In differentiable manifolds, we have the following definition of orientation of a manifold.

Definition 3.9 (Orientability of a Manifold)**.** Suppose M is an n-manifold. Pick any two charts $(U, \phi), (V, \psi)$ of M. Then M is said to be **orientable** if there is a smooth atlas such that the Jacobi matrix of each transition map $\psi \circ \phi^{-1}$ has positive determinant at each point.

Note that the Jacobi matrix of the transition map is just the differential map of the transition map. Therefore, the above definition can be rephrased in terms of the transition maps on the tangent bundle. Then we can say that the tangent bundle TM is orientable if M is orientable. More generally, we have the following definition of the orientability of a real vector bundle, which is equivalent to the condition that M is orientable when we restrict to the case $TM \to M$.

Definition 3.10 (Orientability of a Real Vector Bundle)**.** Suppose $p\colon E \to B$ is a real vector bundle of dimension n. Pick two bundle charts $(U, \phi), (V, \psi)$ for p. Then the transition map gives a map $g\colon U \cap V \to \mathrm{GL}_n(\mathbb{R})$ by

$$\psi \circ \phi^{-1}\colon (U \cap V) \times \mathbb{R}^n \to (U \cap V) \times \mathbb{R}^n, \quad (x, v) \mapsto \big(x, g_x(v)\big)$$

Then p is said to be **orientable** if there is a bundle atlas such that every element in the image of g_x have positive determinant for all x.

In fact, the orientability of a bundle is encoded in the cohomology group.

Proposition 3.11. *Suppose $p\colon E \to B$ is a real vector bundle of dimension n. Let $p'\colon E' \to B$ be the subbundle where E' is E minus the zero section of p. Then p is orientable if and only if there exists a $t \in H^n(E, E'; \mathbb{Z})$ such that t restricts to a generator in $H^n(F_b, F_b'; \mathbb{Z})$ for each $b \in B$, where F_b, F_b' are fibers over b in E, E' respectively.*

Proof. See [TD08, Theorem 17.9.4]. □

We can generalize this to arbitrary generalized cohomology theories associated to some ring spectrum.

Definition 3.12 (E-Orientation)**.** Suppose E is a ring spectrum. Let $p\colon V \to B$ be a vector bundle of dimension n. Then an E**-orientation** on p is an element in $E^n\big(Th(V)\big)$ restricting to a generator in $E^n(S^n) \cong \pi_0(E)$ on each fiber, where $Th(V)$ is the Thom space of V.

Note that all real manifolds are $H\mathbb{Z}/2$-orientable. It inspires us to define the orientability of the generalized cohomology theory itself so that all vector bundles have a canonical choice of orientation. Here we only want to focus on the complex vector bundles.

Definition 3.13 (Complex Orientation)**.** A **complex orientation** on a ring spectrum E is a family of elements $c_V \in E^{2n}\big(Th(V)\big)$ for each $n \in \mathbb{N}$ and complex vector bundle $V \to B$ of dimension n such that

(a) For any $b \in B$, c_V restricts to a generator in $E^{2n}\big(Th(V_x)\big) \cong E^{2n}(S^{2n}) \cong \pi_0(E)$.

(b) For any map $f\colon B' \to B$, $c_{f^*V} = f^*(c_V)$.

(c) For any two complex vector bundles V_1, V_2 over B, $c_{V_1 \oplus V_2} = c_{V_1} \cdot c_{V_2}$.

We know that there is a universal 1-dimensional complex vector bundle γ_1 over $\mathbb{CP}^\infty$.

Theorem 3.14. *A complex orientation is determined by the element $c_{\gamma_1} \in E^2\big(Th(\gamma_1)\big)$. There is a bijection between the elements in $E^2\big(Th(\gamma_1)\big) \cong E^2(\mathbb{CP}^\infty)$ that restricts to 1 in $E^2(S^2) \cong \pi_0(E)$ and complex orientations of E.*

Proof. See [TD08, Theorem 19.0.1]. □

Suppose E is complex oriented. Due to [TD08, Theorem 19.1.4, Proposition 19.1.6], we have $E^*\big(\mathbb{CP}^\infty\big) = E_*[\![T]\!]$ and $E^*\big(\mathbb{CP}^\infty \times \mathbb{CP}^\infty\big) = E_*[\![X, Y]\!]$, where $\deg(T) = \deg(X) = \deg(Y) = 2$ and T is the chosen complex orientation of E. Note that $\mathbb{CP}^\infty \simeq \mathrm{BU}(1)$. Therefore, there is a symmetric multiplication map $m \colon \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$. The induced map on cohomology rings sends T to an element $f(X, Y) \in E_*[\![X, Y]\!]$. By the associativity and commutativity of m, we have

Proposition 3.15. *The above $f(X, Y)$ is a formal group law with coefficients in E_*.*

Different choice of T will generate different formal group laws. We also call T a **coordinate**.

Example 3.16. *The Eilenberg-Maclane spectrum HA is complex oriented, where the $T \in HA^2(\mathbb{CP}^\infty)$ is the first Chern class. Then the formal group law associated to this is the additive formal group law.*

3.3 Complex Cobordism Theory

For each $n \in \mathbb{N}$, let $\mathrm{BU}(n)$ be the classifying space of $U(n)$, the group of unitary matrices of rank n. Let γ_n be the universal complex n-bundle over $\mathrm{BU}(n)$. If we identify $\mathrm{BU}(n)$ as the Grassmanian G_n, i.e., the space of n-dimensional subspaces in $\mathbb{C}^\infty$. The sphere bundle $S(\gamma_n)$ of γ_n consists of pairs (v, W), where W is an n-dimensional subspace in $\mathbb{C}^\infty$ and $v \in W$ is a unit vector. Then we have a map $S(\gamma_n) \to G_{n-1} \simeq \mathrm{BU}(n-1)$ sending (v, W) to the orthogonal complement of v in W. This is a fiber bundle with fiber S^∞, i.e., all the unit vectors in $\mathbb{C}^\infty$. Since S^∞ is contractible, $\mathrm{BU}(n-1) \simeq S(\gamma_n)$, which is homotopy equivalent to the space obtained by γ_n minus the zero section of $BU(n)$. Since $\gamma_n \simeq BU(n)$, $\mathrm{Th}(\gamma_n) \simeq BU(n)/BU(n-1)$. According to [TD08, Theorem 19.3.2], for a complex oriented cohomology theory E, $E^*\big(\mathrm{BU}(n)\big) \cong E_*[\![c_1, \cdots, c_n]\!]$. When $n = 1$, c_1 is just the complex orientation. Therefore, $E^*\big(\mathrm{BU}(n)/\mathrm{BU}(n-1)\big) \cong c_n E_*[\![c_1, \cdots, c_n]\!]$, where $\deg c_i = 2i$.

Let

$$\mathrm{MU}(n) := \Sigma^{-2n}\Sigma^{\infty}\mathrm{Th}(\gamma_n) \simeq \Sigma^{-2n}\Sigma^{\infty}\mathrm{BU}(n)/\mathrm{BU}(n-1)$$

Then c_n is a map $\phi_n\colon \mathrm{MU}(n) \to E$. We have the natural maps

$$\mathrm{MU}(n-1) = \Sigma^{2-2n}\Sigma^{\infty}\mathrm{Th}(\gamma_{n-1}) = \Sigma^{-2n}\Sigma^{\infty}\mathrm{Th}(\gamma_{n-1}\oplus\epsilon) \to \Sigma^{-2n}\Sigma^{\infty}\mathrm{Th}(\gamma_n) = \mathrm{MU}(n)$$

Let $\mathrm{MU} := \mathrm{colim}\mathrm{MU}(n)$, called the **complex cobordism spectrum**. It can be shown that ϕ_n are compatible with the colimit [Lur10, Lecture 6]. Thus, this gives a map $\phi\colon \mathrm{MU} \to E$.

In fact, MU admits a ring structure. Suppose $\gamma_a \oplus \gamma_b$ is classified by $\mathrm{BU}(a) \times \mathrm{BU}(b) \to \mathrm{BU}(a+b)$. It induces a map between Thom spectra $\mathrm{MU}(a) \wedge \mathrm{MU}(b) \to \mathrm{MU}(a+b)$. Passing to the limit we get a ring map $\mathrm{MU} \wedge \mathrm{MU} \to \mathrm{MU}$ with the unit map $S \simeq MU(0) \to MU$. Therefore, MU is a ring spectrum.

Proposition 3.17. *The map ϕ is a map of ring spectra.*

Proof. See [Lur10, Lecture 6, Proposition 6]. □

The inclusion $\Sigma^{-2}\Sigma^{\infty}\mathbb{CP}^{\infty} = \mathrm{MU}(1) \to \mathrm{MU}$ gives an element $T_{\mathrm{MU}} \in \mathrm{MU}^2(\mathbb{CP}^{\infty})$. Since c_1 is just the complex orientation, the ring spectrum map $\phi\colon \mathrm{MU} \to E$ carries T_{MU} to our chosen complex orientation of E.

The induced element T_{MU} is a complex orientation of MU. In fact, the restriction of T_{MU} to S^2 is given by $MU^2(\mathbb{CP}^{\infty}) \to MU^2(S^2)$ induced by $S = \mathrm{MU}(0) \to \mathrm{MU}(1) \to \mathrm{MU}$, which is the unit map of MU. Thus, the restriction of T_{MU} is 1.

Theorem 3.18. *Let E be a ring spectrum. Let $T_{MU} \in MU(\mathbb{CP}^{\infty})$ be a complex orientation of MU. The map $(\phi\colon MU \to E) \to \phi\big(T_{MU}\big)$ constructed above gives a bijection between ring spectra maps $MU \to E$ and complex orientations of E.*

Proof. See [Lur10, Lecture 6, Theorem 8]. □

Therefore, MU is the universal complex oriented generalized cohomology theory.

In fact, MU has a geometric interpretation, which accounts for its name "cobordism". For details and further references, please refer to [Car16].

Definition 3.19 (Complex Oriented Map)**.** Suppose X is a compact smooth manifold. Then a **complex oriented map** to X is a pair (f, ν), where f is a smooth proper map $f\colon M \to X$ such that the relative dimension $\dim f := \dim M - \dim X$ is even and $\nu\colon M \to \mathrm{BU}$ is continuous. In addition, the map f can be factored by

$$M \xrightarrow{i} X \times \mathbb{C}^n \xrightarrow{p} X$$

where i is a topological embedding and p is the natural projection map. The normal bundle of M in $X \times \mathbb{C}^n$ has a complex bundle structure, which is characterized by ν.

A complex oriented map of odd relative dimension is a pair $(f, 0)\colon M \to X \times \mathbb{R}$, where f is a complex oriented map of even relative dimension.

Lemma 3.20. *Suppose $f\colon M \to X$ is complex oriented and $g\colon Y \to X$ is transversal to f. Then the pullback of f along g is also complex oriented.*

Proof. See [Car16, Section 3.1, Pullbacks]. □

We can define an equivalence on complex oriented maps similar to bordism.

Definition 3.21 (Cobordant)**.** Suppose $f_i\colon Z_i \to X$ are two complex oriented maps for $i = 0, 1$. Then f_0, f_1 are said to be **cobordant** if there is a complex oriented map $h\colon W \to X \times \mathbb{R}$ such that h is transversal to maps $j_i\colon X \to X \times \mathbb{R}$ by $x \mapsto (x, i)$ and the pullback of h by each j_i is isomorphic to f_i. This is an equivalent relation [Car16, Definition 3.1.3].

Definition 3.22. For any compact smooth manifold X, we define the following groups

$$U^n(X) := \{(f, \nu)\colon \text{complex oriented maps of relative dimension } n\}/\text{cobordant}$$
$$U^*(X) := \oplus_{n\in\mathbb{Z}} U^n(X)$$

The addition on $U^n(X)$ is given by

$$(f, \nu) + (f', \nu') := (f \sqcup f', \nu \sqcup \nu')$$

We can also define a ring structure on $U^*(X)$ by

$$U^*(X) \times U^*(X) \to U^*(X \times X) \xrightarrow{\Delta^*} U^*(X)$$
$$(f, \nu) \times (f', \nu') \mapsto (f \times f', \nu \times \nu')$$

where Δ is the diagonal map.

Theorem 3.23. *For a compact manifold X,*

$$U^*(X) \cong MU^*(X)$$

given by the Pontrjagin-Thom construction.

Proof. See [Car16, Proposition 3.2.1]. □

3.4 Morava E-Theories

We digress from the topology and come back to formal group laws temporarily. Suppose k is a perfect field of characteristic p and F is a formal group law over k.

Proposition 3.24. *Let R be a commutative ring with characteristic p and F be a formal group law over F. Then either $[p]_F = 0$ or $[p]_F = \lambda T^{p^n} + O(T^{p^n+1})$ for some $n \in \mathbb{N}$ and nonzero $\lambda \in R$, where $[p]_F$ is the p-series of F.*

Proof. See [Lur10, Lecture 12, Proposition 12]. □

Definition 3.25 (Height of a Formal Group Law)**.** Let v_i be the coefficient of T^{p^i} in $[p]_F$ for each i. Say F has **height** n if $v_i = 0$ for $i < n$ and $v_n \neq 0$.

Definition 3.26 (Deformation of a Formal Group Law)**.** Let F be a formal group law over k and A is a complete local ring with the maximal ideal $\mathfrak{m}$ and residue field containing k. Suppose $\pi\colon A \to A/\mathfrak{m}$ is the natural projection and $i\colon k \to A/\mathfrak{m}$ is the inclusion. A **deformation of F to A** is a formal group law $\tilde{F}$ over A, such that $\pi_*(\tilde{F}) = i_*(F)$, where π, i act on each coefficient. Let G, H be two deformations of F over A. Then the two deformations are said to be **$\star$-isomorphic** if there is an isomorphism $\sigma\colon G \to H$ such that $\pi_*(\sigma) = T$. Then define

$$\mathrm{Def}(A, F) := \{\tilde{F} \text{ is a deformation of } F \text{ over } A\}/\star\text{-isomorphic}$$

Let $W(k)$ be the Witt vector over k, which is a complete local ring over with the maximal ideal (p) and residue field k. The precise definition of the Witt vector is too complicated. We just give an example. If $k = \mathbb{F}_q$ where $q = p^n$ for some prime number p, then $W(k)$ is the unique unramified extension of $\mathbb{Z}_p$ of degree n. For references about the Witt vector, one may consult [Rab14]. The following theorem classifies deformations of F.

Theorem 3.27 (Lubin-Tate)**.** *For any formal group law F of height n over k, there is a universal formal group law Γ over $\mathscr{R} := W(k)[\![v_1, \cdots, v_{n-1}]\!]$ such that for any complete local ring A with residue field containing k, there is a bijection*

$$\begin{aligned} Hom_{/k}(\mathscr{R}, A) &\to Def(A, F) \\ \phi &\mapsto \phi_*(\Gamma) \end{aligned}$$

Furthermore, v_i is the coefficient of T^{p^i} in $[p]_\Gamma$.

Proof. See [Lur10, Lecture 21, Theorem 5 and Remark 8]. □

Recall that a complex oriented generalized cohomology theory gives a formal group law. A natural converse question is that given a formal group law over a ring, is there a generalized cohomology has the same coefficient ring and formal group law? The answer is given by the Landweber exact functor theorem.

Theorem 3.28 (Landweber Exact Functor Theorem)**.** *Let F be a formal group law over a commutative graded ring R. Let p be a prime number and v_i be the coefficient of T^{p^i} in $[p]_F$. If $v_0, \cdots, v_i$ forms a regular sequence, i.e., v_i is not a zero-divisor in $R/(v_0, \cdots, v_{i-1})$, for all i and p, then there is a homology theory E such that $E_* = R$ and the associated formal group law is F.*

Proof. See [Lur10, Lecture 16, Theorem 1]. □

Remark. *Recall that Brown representability theorem only applies to cohomology theory. However, when restricted to finite CW-complexes, it also works for homology theories using Spanier-Whitehead duality [Rav92, Section 5.2]. Therefore, we obtain a spectrum representing the homology theory (over finite CW-complexes).*

We want to apply the theorem to the universal deformation Γ over $\mathscr{R}$. For the prime number $p = \text{char}(k)$, $(v_0 = p, v_1, \cdots, v_{n-1})$ is a maximal ideal of $\mathscr{R}$ and v_n is invertible in $k = \mathscr{R}/(v_0, \cdots, v_n)$ since F has height n. For a prime number $p' \neq p$, p' is invertible in $\mathscr{R}$, so $\mathscr{R}/p' = 0$. Therefore, Γ and $\mathscr{R}$ satisfy the condition of Landweber exact functor theorem.

Definition 3.29 (Morava E-Theory)**.** The generalized cohomology theory associated to the universal formal group law over $\mathscr{R}[\beta^{\pm 1}]$ with $\deg(\beta) = 2$ is called **Morava E-theory** E_n, which is also called **Lubin-Tate theory**.

Remark. *Morava E-theory plays an important role in chromatic homotopy theory. There is an analogy of localization of rings in topology called Bousfield localization, through which we can localize a space with respect to some spectrum. The localization with respect to Morava E-theory stands for formal group laws with height $\leqslant n$. Furthermore, the homotopy fixed points of E_n under the action of a certain group is homotopy equivalent to the localization of the sphere spectrum with respect to Morava K-theory $K(n)$, which is another important spectrum in chromatic homotopy theory. The latter localization is essential in the computation of stable homotopy groups. For detailed references in chromatic homotopy theory, see [Rav92] and [Lur10].*

Remark. *There are several terms involving "Lubin-Tate". The first is the Lubin-Tate formal group laws, which are important tools in the proof of explicit local class field theory as shown in Section 2. The second is the Lubin-Tate theory, which is the theory of deformation of formal group laws, i.e., Theorem 3.27. The third is the Morava E-theory above. The latter two terms share the same name. Sometimes it is quite confusing.*

There is some relationship between the three terms. Suppose K is a local field with residue field k with characteristic $p > 0$ and $|k| = q$. Then Lubin-Tate formal group laws are the lifting of formal group laws F over k such that $[p]_F = T^q$, so that they can be classified by Theorem 3.27. On the other hand, the construction of Lubin-Tate spectrum is based on the Lubin-Tate theory (of deformation) as shown above.

3.5 H_∞-Maps and Power Operations

Definition 3.30 (H_∞-Ring Spectrum and H_∞-Map)**.** A ring spectrum E that is a commutative monoid in the stable homotopy category is called an H_∞**-ring spectrum**. Morphisms between H_∞ spectra are called H_∞**-maps**.

Remark. *If E is a commutative monoid in the stable category, we can replace H_∞ by E_∞.*

Example 3.31. *The complex cobordism theory MU is E_∞ [May77, §IV.2]. Morava E-theories are E_∞ [GH04, Corollary 7.6].*

Power operation is an important structure on cohomology theories. It is a refinement of taking powers in cohomology rings. The total power operation is of the form $P_m\colon E^0(X) \to E^0(X \times B\Sigma_m)$, where E is a cohomology theory, X is a spectrum and $B\Sigma_m$ is the classifying space of the symmetric group of m elements. Actually, m-th power on $E^0(X)$ factors through P_m. If a spectrum is H_∞, then it admits a power operation structure. Moreover, for two H_∞-spectra E, F, ring spectra morphisms such that power operations are compatible are equivalent to H_∞-maps. By compatible, we mean that for a ring spectra morphism $f\colon E \to F$, the diagram

$$\begin{array}{ccc} E^0(X) & \xrightarrow{P_m^E} & E^0(X \times B\Sigma_m) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ F^0(X) & \xrightarrow[P_m^F]{} & F^0(X \times B\Sigma_m) \end{array}$$

commutes. Details can be found in [BMMS86].

4 Proof of Ando's Theorem via Coleman Norm Operators

4.1 Coleman Norm Operators

Let $q = p^n$ and $k = \mathbb{F}_q$. Suppose K is the unramified extension of $\mathbb{Q}_p$ of degree n with maximal integer ring $\mathcal{O}_K$, maximal ideal $\mathfrak{m} = \pi\mathcal{O}_K$ and residue field k. Thus, p is an uniformizer of K.

Suppose $\mathcal{O}_K(\!(T)\!)$ is the ring of Laurent series with coefficients in $\mathcal{O}_K$. We assign the "compact-open" topology to $\mathcal{O}_K(\!(T)\!)$, i.e., a sequence $\{g_n\}$ converges to g if and only if for any compact subset A not containing 0 in $\mathfrak{m}$, and for each $\epsilon > 0$, there exists a positive integer $N = N(A, \epsilon)$ such that $|g_n(a) - g(a)| < \epsilon$ for all $a \in A$ and $n \geqslant N$. If g_n converge to g, then they converge on each term. Then Coleman norm operator is given by:

Theorem 4.1. *There exists a unique $\mathcal{N}_{F_f}\colon \mathcal{O}_K(\!(T)\!) \to \mathcal{O}_K(\!(T)\!)$ satisfying*

$$\mathcal{N}_{F_f}(g) \circ [p]_{F_f} = \prod_{\lambda \in \Lambda_{f,1}} g(T +_{F_f} \lambda)$$

for every $g \in \mathcal{O}_K(\!(T)\!)$. Moreover, $\mathcal{N}_{F_f}$ is continuous and multiplicative.

Proof. See [Col79, Theorem 11, Corollary 12]. □

The norm operator has the following properties.

Lemma 4.2. *Let $i \geqslant 1$, $g \in 1 + \mathfrak{m}^i[\![T]\!]$ and h is a unit in $\mathcal{O}_K(\!(T)\!)$. Then*

(a) $\mathcal{N}_{F_f}(g) \in 1 + \mathfrak{m}^{i+1}[\![T]\!]$.

(b) $\mathcal{N}_{F_f}^i(h)/\mathcal{N}_{F_f}^{i-1}(h) \in 1 + \mathfrak{m}^i[\![T]\!]$.

Proof. See [Col79, Lemma 13]. The part (b) looks different from [Col79, Lemma 13(b)], which said $\mathscr{N}_{F_f}^i(h)/\phi\mathscr{N}_{F_f}^{i-1}(h) \in 1+\pi^i\mathcal{O}_K[\![T]\!]$. Because Coleman generalized the construction of the norm operator to a complete unramified extension H/K, he needed to take the Frobenius map ϕ of $\mathrm{Gal}(H/K)$ into consideration. However, we only need to consider K itself, so $\phi = Id_K$ here. ☐

Then we see that $\mathscr{N}_{F_f}^\infty(h) := \lim_{i\to\infty} \mathscr{N}_{F_f}^i(h)$ exists. By Lemma 4.2(a), $\mathscr{N}_{F_f}^\infty\big(1+\mathfrak{m}[\![T]\!]\big) = 1$. Since $\mathscr{N}_{F_f}$ is continuous,

$$\mathscr{N}_{F_f}\big(\mathscr{N}_{F_f}^\infty(h)\big) = \mathscr{N}_{F_f}\big(\lim_{i\to\infty} \mathscr{N}_{F_f}^i(h)\big) = \lim_{i\to\infty} \mathscr{N}_{F_f}\big(\mathscr{N}_{F_f}^i(h)\big) = \mathscr{N}_{F_f}^\infty(h)$$

Moreover, $\mathscr{N}_{F_f}^\infty$ is multiplicative since $\mathscr{N}_{F_f}$ is.

4.2 Proof of Ando's Theorem in a Special Case

Let $\Phi(T)$ be the Honda formal group law over k of height n, i.e., $[p]_\Phi(T) = T^q$, where $[p]_\Phi(T)$ is the p-series of Φ. Suppose $\pi = p$. For any $f \in \mathcal{F}_\pi$, F_f is a Lubin-Tate formal group law and $[p]_{F_f}(T) = [\pi]_{f,f}(T) = f(T)$ by Proposition 2.12. Thus, F_f is a lifting of Φ. Conversely, every lifting of Φ to $\mathcal{O}_K$ has p-series in $\mathcal{F}_\pi$, so it is a Lubin-Tate formal group law.

Given a complex oriented cohomology theory E, then there is a map between ring spectra $\mathrm{MU} \to E$ by Theorem 3.18. One may ask whether the power operation are compatible under such map. When $E = E_n$, Ando gave a criterion on when the power operations of MU, E_n are compatible under the map $\mathrm{MU} \to E_n$ in terms of the formal group law associated to the map [And95, Theorem 4].

Theorem 4.3 (Ando)**.** *Suppose $k = \mathbb{F}_p$. In each $\star$-isomorphism class of lifting of Φ to the complete local ring $\mathscr{R} = W(k)[\![v_1, \cdots, v_{n-1}]\!][u^\pm]$, there is a unique formal group law F satisfying*

$$[p]_F(T) = \prod_{\lambda\in\Lambda_F} (T +_F \lambda)$$

where Λ_F is the kernel of $[p]_F$.

Remark. *In the age of Ando, E_n classified the Honda formal group law of height n over $k = \mathbb{F}_p$. Nowadays, we define E_n in the way shown in Subsection 3.4.*

Since $\mathscr{R}$ classifies deformation of a formal group law, we expect such statement holds for arbitrary complete local ring. In fact, we will prove

Theorem 4.4. *Suppose l is a perfect field of characteristic p and Φ is the Honda formal group law of height n over l, i.e., $[p]_\Phi = T^{p^n}$. In each $\star$-isomorphism class of lifting of Φ to a complete local domain R with residue field containing l such that $p \neq 0$ in R, there is a unique formal group law F satisfying*

$$[p]_F(T) = \prod_{\lambda \in \Lambda_F} (T +_F \lambda) \tag{1}$$

where Λ_F is the kernel of $[p]_F$.

Remark. *Here we require $p \neq 0$ in R because we need $[p]_F$ to be able to be canceled in composition and multiplication. Note that the ring $(E_n)_*$ satisfies the condition.*

Remark. *Actually, [Zhu20, Theorem 1.2] proved a more general statement for not only Honda formal group law, but also arbitrary formal group law of finite height over l and R can be any complete local ring with residue field containing l. However, we will only prove the relative specific version in this thesis.*

We will prove the theorem in a special case in this subsection via Coleman norm operator.

Theorem 4.5 (Ando, Special Case)**.** *In each $\star$-isomorphism class of lifting of Φ to $\mathcal{O}_K$, there is a unique formal group law F_f satisfying*

$$[p]_{F_f}(T) = \prod_{\lambda \in \Lambda_{f,1}} (T +_{F_f} \lambda)$$

In terms of the norm operator, we see that a Lubin-Tate formal group law satisfies (1) if and only if

$$[p]_{F_f}(T) = \prod_{\lambda \in \Lambda_{f,1}} (T +_{F_f} \lambda) =: \big(\mathcal{N}_{F_f}(T) \circ [p]_{F_f}\big)(T)$$

Since $[p]_{F_f}(T)$ has a composition inverse in $K[\![T]\!]$, we can cancel the f from both sides, so that (1) is equivalent to

$$\mathcal{N}_{F_f}(T) = T$$

Fix a lifting F_f of Φ. Pick $u \in T + \pi T\mathcal{O}_K[\![T]\!] = T + T\mathfrak{m}[\![T]\!]$. Then there is an $f_u \in \mathcal{F}_\pi$ such that $u \circ F_f \circ u^{-1} = F_{f_u}$. Since $f = [p]_{F_f}$ and $f_u = [p]_{F_{f_u}}$, $f_u = u \circ f \circ u^{-1}$ and $F_{f_u} = F_{u \circ f \circ u^{-1}}$. By the above discussion, we are reduced to showing that there is a unique $u \in T + T\mathfrak{m}[\![T]\!]$ such that

$$\mathcal{N}_{F_{f_u}}(T) = T$$

Note that u induces a bijection from $\Lambda_{f,1}$ to $\Lambda_{f_u,1}$. By definition,

$$\big(\mathscr{N}_{F_{f_u}}(T)\circ[p]_{F_{f_u}}\big)(T)=\prod_{\lambda\in\Lambda_{f_u,1}}(T+_{F_{f_u}}\lambda)$$

This is equivalent to

$$\begin{aligned}\big(\mathscr{N}_{F_{f_u}}(t)\circ u\circ[p]_{F_f}\circ u^{-1}\big)(T)&=\prod_{\lambda\in\Lambda_{f,1}}\big(T+_{F_{f_u}}u(\lambda)\big)\\&=\prod_{\lambda\in\Lambda_{f,1}}F_{f_u}\Big(u\big(u^{-1}(T)\big),u(\lambda)\Big)\\&=\prod_{\lambda\in\Lambda_{f,1}}u\circ F_f\big(u^{-1}(T),\lambda\big)\\&=\prod_{\lambda\in\Lambda_{f,1}}u\circ\big(u^{-1}(T)+_{F_f}\lambda\big)\\&=\big(\mathscr{N}_{F_f}(u)\circ[p]_{F_f}\big)\big(u^{-1}(T)\big)\end{aligned}$$

By canceling $[p]_{F_f}\circ u^{-1}$ from both sides, $\big(\mathscr{N}_{F_{f_u}}(T)\circ u\big)(T)=\mathscr{N}_{F_f}(u)(T)$. Therefore,

$$\mathscr{N}_{F_{f_u}}(T)=T\Leftrightarrow\mathscr{N}_{F_f}(u)=u$$

Consequently, it remains to show the following.

Proposition 4.6. *Given any $f\in\mathcal{F}_\pi$, there is a unique $u\in T+T\mathfrak{m}[\![T]\!]$, such that $\mathscr{N}_{F_f}(u)=u$.*

Proof.

Existence: Suppose $f_i:=\mathscr{N}^i_{F_f}(T)/\mathscr{N}^{i-1}_{F_f}(T)\in1+\mathfrak{m}^i[\![T]\!]$. Then $\mathscr{N}^\infty_{F_f}(T)=Tf_1f_2\cdots$. It is easy to see that $f_1f_2\cdots\in1+\mathfrak{m}[\![T]\!]$, so $\mathscr{N}^\infty_{F_f}(T)\in T+T\mathfrak{m}[\![T]\!]$. Therefore, $u=\mathscr{N}^\infty_{F_f}(T)$ satisfies the condition.

Uniqueness: If $\mathscr{N}_{F_f}(u)=u$, then $\mathscr{N}^i_{F_f}(u)=u$ for each i. Thus, $\mathscr{N}^\infty_{F_f}(u)=u$ after taking the limit. Since $u\in T+T\mathfrak{m}[\![T]\!]$, there is $\tilde{u}\in1+\mathfrak{m}[\![T]\!]$ such that $u=T\tilde{u}$. Then

$$u=\mathscr{N}^\infty_{F_f}(u)=\mathscr{N}^\infty_{F_f}(T)\mathscr{N}^\infty_{F_f}(\tilde{u})=\mathscr{N}^\infty_{F_f}(T)$$

which finishes the proof. □

Remark. *The condition $\mathscr{N}_F(u)=u$ is equivalent to say that u is norm-coherent in the sense of [Col79]. To be precise, suppose v_n is a generator of $\Lambda_{f,n}$ as a $\mathcal{O}_K$-module and $[p]_F(v_{n+1})=v_n$.*

We have

$$\mathscr{N}_F(u)(v_n) = N_{K_{\pi,n+1}/K_{\pi,n}}\big(u(v_{n+1})\big)$$

by [Col79, Corollary 12(ii)]. Thus, $\mathscr{N}_F(u) = u$ is equivalent to say that

$$u(v_n) = N_{K_{\pi,n+1}/K_{\pi,n}}\big(u(v_{n+1})\big)$$

That is, u maps the sequence v_n to a norm coherent sequence.

Suppose $\mathscr{M}_\infty = \{g \in \mathcal{O}_K((T))^* : \mathscr{N}_F(g) = g\}$ is the subset in $\mathcal{O}_K((T))^*$ consisting of norm-coherent series. Then the uniqueness of u is a consequence of the exact sequence of groups:

$$1 \to 1 + \mathfrak{m}[\![T]\!] \to \mathcal{O}_K((T))^* \overset{\mathscr{N}_F^\infty}{\to} \mathscr{M}_\infty \to 1$$

[Col79, Proposition 14].

4.3 Generalization of the Norm Operators

In this subsection, we aim to prove Theorem 4.4 following the proof in the last subsection. Observe that the proof in Subsection 4.2 actually does not use the properties of $\mathcal{O}_K$ being a complete discrete valuation ring with uniformizer p. Therefore, we only need to generalize Theorem 4.1 and Lemma 4.2 to R.

Suppose F is a lifting of Φ to R and $\mathfrak{m}$ is the maximal ideal of R. Since $[p]_F \equiv T^{p^n} \pmod{\mathfrak{m}}$, not all coefficients of $[p]_F$ are in $\mathfrak{m}$. By the Weierstrass preparation theorem [Lan02, Chapter IV, Theorem 9.2], there is a unit v in $R[\![T]\!]$ and a monic polynomial $\beta(T) = T^s + b_{s-1}T^{s-1} + \cdots + b_0$, where $b_i \in \mathfrak{m}$ for all i, such that $[p]_F = v \cdot \beta$. Then the coefficient of T^s in $[p]_F$ is not in $\mathfrak{m}$. Therefore, $s = p^n$. Note that roots of $[p]_F$ are the same with the roots of β. Let Λ be the set of roots of $[p]_F$, which is a finite subset of a larger ring $\tilde{R}$ obtained by R adjoining roots of β. Since $p \neq 0$ in R, 0 is a simple root of $[p]_F$. For any $\lambda \in \Lambda$, $[p]_F\big(T -_F \lambda\big) = [p]_F(T)$. Therefore, λ is also a simple root of $[p]_F$. Thus, roots of $[p]_F$ are distinct in $\tilde{R}$. Therefore, the set Λ has exactly p^n elements. The following proofs basically follow the corresponding proofs in [Col79].

Lemma 4.7. *If $g \in R[\![T]\!]$ and $g(T +_F \lambda) = g(T)$ for all $\lambda \in \Lambda$, then there is a unique $h \in R[\![T]\!]$ such that $h \circ [p]_F = g$.*

Proof. The uniqueness follows from that fact that $[p]_F$ can be canceled.

Let $g_0 = g$. Suppose that we have constructed $a_i \in R$ for $0 \leqslant i \leqslant m-1$ such that

$$g - \sum_{i=0}^{m-1} a_i [p]_F^i = [p]_F^m \cdot g_m$$

for some $g_m \in R[\![T]\!]$. Note that $g(T +_F \lambda) = g(T)$ and $[p]_F(T +_F \lambda) = [p]_F(T)$. We have $g_m(T +_F \lambda) = g_m(T)$. Therefore, $\big(g_m - g_m(0)\big)(\lambda) = 0$ for all $\lambda \in \Lambda$. By [Lan02, Chapter IV, Theorem 9.1], there is a $g_{m+1} \in R[\![T]\!]$ and $r_m \in R[T]$ such that $g_m - g_m(0) = [p]_F \cdot g_{m+1} + r_m$ and $\deg(r_m) < p^n$. Then r_m vanishes on Λ. Since Λ has p^n elements, $r_m = 0$. Let $a_m = g_m(0)$. Then

$$g - \sum_{i=0}^{\infty} a_i [p]_F^i \in \bigcap_{i=0}^{\infty} [p]_F^i R[\![T]\!] = 0$$

Then $h = \sum_{i=0}^{\infty} a_i T^i$ is the required element. □

Now we also give $R[\![T]\!]$ the compact-open topology similar to $\mathcal{O}_K[\![T]\!]$. Here R is assigned with the $\mathfrak{m}$-adic topology.

Theorem 4.8. *There is a unique operator $\mathscr{N}_F \colon R[\![T]\!] \to R[\![T]\!]$ such that for any $g \in R[\![T]\!]$,*

$$\mathscr{N}_F(g) \circ [p]_F(T) = \prod_{\lambda \in \Lambda} g(T +_F \lambda)$$

Moreover, $\mathscr{N}$ is multiplicative and continuous.

Proof. Note that the right hand satisfies the condition of last lemma. Thus, there is a unique $\mathscr{N}_F$ satisfying the formula.

For any $g, h \in R[\![T]\!]$,

$$\begin{aligned} \mathscr{N}_F(gh) \circ [p]_F(T) &= \prod_{\lambda \in \Lambda} gh(T +_F \lambda) \\ &= \big(\mathscr{N}_F(g) \circ [p]_F(T)\big) \cdot \big(\mathscr{N}_F(h) \circ [p]_F(T)\big) \\ &= \big(\mathscr{N}_F(g) \cdot \mathscr{N}_F(h)\big) \circ [p]_F(T) \end{aligned}$$

Canceling $[p]_F$ from both sides we get $\mathscr{N}_F(gh) = \mathscr{N}_F(g) \cdot \mathscr{N}_F(h)$.

Suppose $\{g_n\}$ converges to g.

$$\begin{aligned}\big(\lim \mathscr{N}_F(g_n)\big) \circ [p]_F = \lim\big(\mathscr{N}_F(g_n) \circ [p]_F\big) &= \lim \prod_{\lambda\in\Lambda} g_n(T +_F \lambda) \\ &= \prod_{\lambda\in\Lambda} g(T +_F \lambda) = \mathscr{N}_F(g) \circ [p]_F\end{aligned}$$

By canceling $[p]_F$ from each side, we get $\lim \mathscr{N}_F(g_n) = \mathscr{N}_F(g)$. □

Remark. *Lemma 4.7 may fail when $p = 0$ in R. Suppose $R = \mathbb{F}_p[\![T]\!]$ and F is just the Honda formal group law. Then $\Lambda = \{0\}$. Thus, for any $g \in R[\![T]\!]$, $g(T +_F \lambda) = g(T)$ for all $\lambda \in \Lambda$. Then the lemma is equivalent to say that $[p]_F = T^{p^n}$ is invertible in composition, which is ridiculous.*

However, the norm operator still exists. Now the condition reads

$$\mathscr{N}_F(g)(T^{p^n}) = g^{p^n}(T)$$

Thus, $\mathscr{N}_F(g)$ is the power series obtained from g such that each coefficient of $\mathscr{N}_F$ is the p^n-th power of the corresponding coefficient in g.

Note that the proof in subsection 4.2 only takes the limit of $\mathscr{N}_F$ on $1 + \mathfrak{m}[\![T]\!]$ and T.

Lemma 4.9. *Let $g \in 1 + \mathfrak{m}^i[\![T]\!]$ and $i \geqslant 1$. Then*

(a) $\mathscr{N}_F(g) \in 1 + \mathfrak{m}^{i+1}[\![T]\!]$.

(b) $\mathscr{N}_F^i(T)/\mathscr{N}_F^{i-1}(T) \in 1 + \mathfrak{m}^i[\![T]\!]$.

Proof. (a) By definition, $\mathscr{N}_F(g) \circ [p]_F = \prod_{\lambda\in\Lambda} g(T +_F \lambda)$. Suppose $g(T) = 1 + \sum_{j=0}^{\infty} c_j T^j$, where $c_j \in \mathfrak{m}^i$. Since $i \geqslant 1$, terms containing $c_{j_1}c_{j_2}$ must lie in $\mathfrak{m}^{i+1}$. Therefore,

$$\begin{aligned}\mathscr{N}_F(g) \circ [p]_F &\equiv 1 + \sum_{\lambda\in\Lambda}\sum_{j=0}^{\infty} c_j (T +_F \lambda)^j \pmod{\mathfrak{m}^{i+1}} \\ &= 1 + \sum_{j=0}^{\infty}\sum_{\lambda\in\Lambda} c_j (T +_F \lambda)^j \\ &= 1 + \sum_{j=0}^{\infty} c_j \big(p^n T^j + \sum_{k=0}^{\infty} p_k(\Lambda) T^k\big)\end{aligned}$$

where $p_K(\Lambda)$ is a symmetric function on $\lambda \in \Lambda$. By [Art91, Theorem 16.1.6], $p_k(\lambda)$ is a polynomial of non-leading coefficients in β, i.e., $b_0, \cdots, b_{s-1}$. Since $p^n, b_0, \cdots, b_{s-1}$ are in $\mathfrak{m}$,

$$\mathcal{N}_F(g) \circ [p]_F \equiv 1 \pmod{\mathfrak{m}^{i+1}}$$

Next we prove by induction on i that if $h \in R[\![T]\!]$ and $h \circ [p]_F \in \mathfrak{m}^i[\![T]\!]$, then $h \in \mathfrak{m}^i[\![T]\!]$ (here $i \geqslant 0$). Taking $h = \mathcal{N}_F(g) - 1$ completes the proof of (a). The case is trivial when $i = 0$. Suppose $i \geqslant 1$ and the statement holds for $i - 1$. By the induction hypothesis, $h \in \mathfrak{m}^{i-1}[\![T]\!]$. Suppose $h(T) = \sum_{j=0}^{\infty} d_j T^j$, where $d_j \in \mathfrak{m}^{i-1}$. If $\{j : d_j \notin \mathfrak{m}^i\}$ is non-empty, let j_0 be the minimal number in $\{j : d_j \notin \mathfrak{m}^i\}$. Suppose $[p]_F = \sum_{j=0}^{\infty} a_j T^j$. Since Φ is of height n, $[p]_F \equiv a_{p^n} T^{p^n} + O(T^{p^n+1}) \pmod{\mathfrak{m}}$. Thus,

$$d_{j_0}[p]_F^{j_0} \equiv d_{j_0} a_{p_n} T^{j_0 p^n} + O(T^{j_0 p^n + 1}) \pmod{\mathfrak{m}^i}$$

where a_{p_n} is invertible in R. Since $h \circ [p]_F \in \mathfrak{m}^i[\![T]\!]$, there is a non-negative integer $m \neq j_0$ such that $d_m [p]_F^m$ contains a term with coefficient in $\mathfrak{m}^{i-1} - \mathfrak{m}^i$ at degree $j_0 p^n$. If $m < j_0$, then $d_m \in \mathfrak{m}^i$ by the minimality of j_0, contradiction. If $m > j_0$, suppose the term is $d_m a_{j_1} a_{j_2} \cdots a_{j_m} T^{j_1 + j_2 + \cdots + j_m}$, where $j_1 + j_2 + \cdots + j_m = j_0 p^n$. Since $m > j_0$, there must be a $j_k < p^n$. Then $a_{j_k} \in \mathfrak{m}$, contradiction. Therefore, $h \in \mathfrak{m}^i[\![T]\!]$.

(b) By (a), we only need to show that case when $i = 1$. Since $[p]_F \equiv T^{p^n} \pmod{\mathfrak{m}}$,

$$\mathcal{N}_F(T^{p^n}) \equiv \mathcal{N}_F \circ [p]_F(T) = \prod_{\lambda \in \Lambda} (T +_F \lambda) \pmod{\mathfrak{m}}$$

By arguments similar to (a), $\prod_{\lambda \in \Lambda} (T +_F \lambda) \equiv T^{p^n} \pmod{\mathfrak{m}}$. Hence, $\mathcal{N}_F(T) \equiv T \pmod{\mathfrak{m}}$, so $\mathcal{N}_F(T)/T \equiv 1 \pmod{T^{-1}\mathfrak{m}[\![T]\!]}$. It remains to show that $T \mid \mathcal{N}_F(T)$ in $R[\![T]\!]$. It is equivalent to say that $\mathcal{N}_F(T)(0) = 0$. Since $0 \in \Lambda$,

$$\mathcal{N}_F(T)(0) = \mathcal{N}_F(T) \circ [p]_F(0) = \prod_{\lambda \in \Lambda} \lambda = 0$$

□

Remark. *In the proof of (a), we do not require that Φ is a Honda formal group law. We just need Φ to be of height $n < \infty$.*

However, Part (b) of the last lemma may not be true when Φ is not a Honda formal group *law. Suppose* Φ *has height* n*. Note that*

$$\mathscr{N}_F([p]_\Phi) \equiv T^{p^n} \pmod{\mathfrak{m}}$$

Suppose $\mathscr{N}_F(T) = \sum c_j T^j$ *and* $[p]_\Phi = \sum a_j T^j$*. Then by direct calculation we find that* $c_2 \equiv -a_{p^n}^{-3} a_{2p^n} \pmod{\mathfrak{m}}$ *may not be zero.*

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